Oscillation modes of thick accretion disks using finite-elements calculations

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Outline

- Stationary configurations
  → Construction in general case
  → What limits the size?
- Linear perturbations
  → Relativistic Papaloizou-Pringle equation
  → Constant angular momentum
  → Analytical calculation (overview)
- Numerical calculations using FEM
  → Weak formulation of the problem
  → Implementation
- Results for axisymmetric modes
  → Comparison with the analytic results (eigenfrequencies)
  → Comparison of eigenfunctions
- Non-axisymmetric modes
  → Mode merging and Papaloizou-Pringle instability
- Discussion and conclusions

EQUILIBRIUM

Equilibrium model: analytic theory

- Perfect fluid:
$$T^{\alpha}_{\beta} = (e+p)u^{\alpha}u_{\beta} + p\delta^{\alpha}_{\beta}$$
- Pure rotation:
$$u^{\mu} = A\left(t^{\mu} + \Omega\phi^{\mu}\right), \quad u_{\nu} = -E\left(\delta^{t}_{\nu} - \ell\delta^{\phi}_{\nu}\right).$$
- Conservation laws:
$$\nabla_{\alpha}\left(\rho u^{\alpha}\right) = 0, \quad \nabla_{\alpha}T^{\alpha}_{\beta} = 0.$$
$$(e+p)\boldsymbol{a} + \boldsymbol{\nabla} p = 0, \quad \boldsymbol{a} = \boldsymbol{\nabla}\ln E - \frac{\Omega\boldsymbol{\nabla}\ell}{1-\ell\Omega}.$$
- This can be integrated if $p = p(n)$, $\rho = \rho(n)$ to
$$\boxed{E\Psi h = \mathrm{const}} \quad \Psi \equiv \exp\left[-\int\frac{\Omega d\ell}{1-\ell\Omega}\right].$$

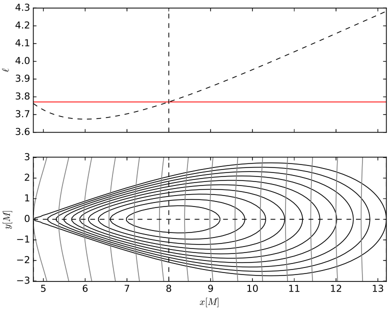
Equilibrium model: construction

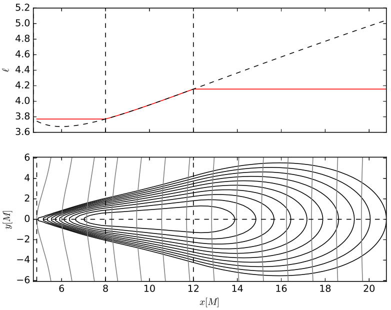
Von Zeipel cylindres:

$$\Omega = \Omega(\ell) \quad \text{and} \quad \Omega = \frac{g^{\phi t} - \ell g^{\phi\phi}}{g^{tt} - \ell g^{t\phi}}.$$

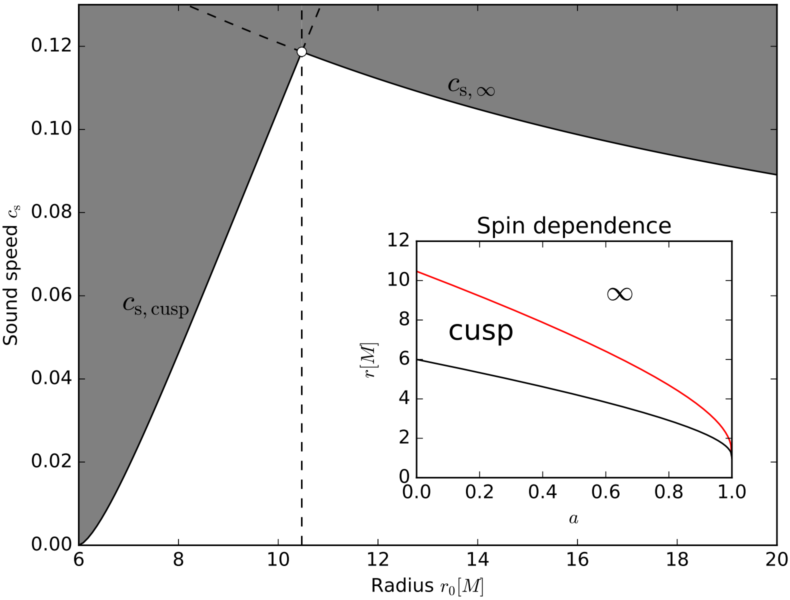
Lane-Embden function (polytropes):

$$f = 1 - \frac{1}{N c_{\mathrm{s0}}^2}\left(1 - \frac{E_0 \Psi_0}{E \psi}\right), \quad 0 \leq f \leq 1.$$

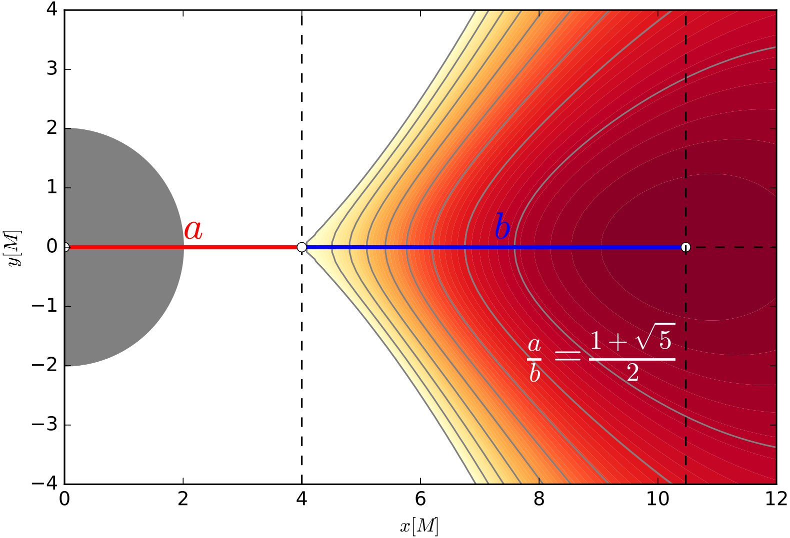




What limits torus sizes?



Maximal torus in Schwarzschild spacetime



Perturbations

Perturbations

- Perturbation of conservation laws

$$\nabla_\alpha \delta\left(\rho u^\alpha\right) = 0, \quad \nabla_\alpha \delta T^\alpha_\beta = 0.$$

Introduce

$$\tilde{\omega} \equiv \omega - m\Omega, \quad \tilde{m} \equiv m - \ell\omega, \quad \eta \equiv \frac{\delta h}{h}$$

$$\begin{aligned} -\mathrm{i}\tilde{\omega}\delta\boldsymbol{u} + \boldsymbol{\gamma}_3 \delta u_\phi + \frac{1}{A}\boldsymbol{\nabla}\eta &= 0, \\ -\mathrm{i}\tilde{\omega}\delta u_\phi + \boldsymbol{\gamma}_1 \cdot \delta\boldsymbol{u} + \mathrm{i}\tilde{m}E\eta &= 0, \\ \frac{1}{\rho R}\boldsymbol{\nabla}\cdot(\rho R\delta\boldsymbol{u}) + \frac{\mathrm{i}\tilde{m}E}{AR^2}\delta u_\phi - \frac{\mathrm{i}\tilde{\omega}A}{c_\mathrm{s}^2}\eta &= 0. \end{aligned}$$

- 3 poloidal vectors:

$$\boldsymbol{\gamma}_1 \equiv E^2\boldsymbol{\nabla}\ell, \quad \boldsymbol{\gamma}_2 \equiv \boldsymbol{\nabla}\Omega, \quad \boldsymbol{\gamma}_3 \equiv \boldsymbol{\gamma}_2 - \frac{\boldsymbol{\gamma}_1}{A^2R^2}.$$

Papaloizou-Pringle equation

Perturbation variable

$$W \equiv -\frac{\delta h}{A\tilde{\omega}} = -\frac{\delta p}{A\tilde{\omega}\rho}.$$

The Euler equation gives

$$\delta \boldsymbol{u} = \frac{\mathrm{i}}{h}\left[\mathbf{P}\cdot\boldsymbol{\nabla}W + \boldsymbol{\gamma}_1\frac{\tilde{m}\tilde{\omega}E}{ADR^2}W\right],$$

$$\delta u_\phi = -\frac{1}{hD}\left[\tilde{\omega}\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}W - \tilde{m}AE\left(\tilde{\omega}^2 + \boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_2\right)W\right],$$

with

$$D \equiv \kappa^2 - \tilde{\omega}^2, \quad \kappa^2 \equiv -\boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_3, \quad \mathbf{P} \equiv \tilde{\mathbf{g}} + \frac{1}{D}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_3.$$

Papaloizou-Pringle equation

Continuity equation gives

$$\frac{h}{\rho R}\nabla\cdot\left(\frac{\rho R}{h}\mathbf{P}\cdot\nabla W\right)+\left[\frac{A\tilde{\omega}}{\rho R}\nabla\cdot\left(\frac{\rho\tilde{m}E}{DRA^2}\gamma_1\right)+\frac{\tilde{m}^2\tilde{\omega}^2E^2}{DR^2}+\frac{A^2\tilde{\omega}^2}{c_s^2}\right]W=0.$$

For constant angular momentum tori:

$$\boxed{\frac{h}{\rho R}\nabla\cdot\left(\frac{\rho R}{h}\nabla W\right)+\left[\frac{A^2\tilde{\omega}^2}{c_s^2}-\frac{\tilde{m}^2E^2}{R^2}\right]W=0,}$$

$$\tilde{\omega}\equiv\omega-m\Omega,\quad \tilde{m}\equiv m-\ell\omega$$

quadratic eigenvalue problem:

$$\boxed{\hat{L}W+(\omega-m\Omega_1)(\omega-m\Omega_2)W=0,}$$

$$\hat{L}\equiv\frac{h}{\rho R\mathcal{B}}\nabla\cdot\left(\frac{\rho R}{h}\nabla\right),\quad \mathcal{B}\equiv\frac{A^2}{c_s^2}-\frac{E^2\ell^2}{R^2},\quad \Omega_{1,2}\equiv\frac{AR\Omega\pm c_sE}{AR\pm c_sE\ell}.$$

Approximate solution for not so large tori

- For infinitely small tori ($\beta \to 0$), the PP equation has a *simple* form:

$$\boxed{\hat{L}^{(0)} W^{(0)} - \tilde{\omega}^{(0)2} W = 0}, \quad \tilde{\omega}^{(0)} \equiv \omega - m\Omega_0 = \text{const.}$$

... linear eigenvalue problem for $(\tilde{\omega}^{(0)}, W)$

- The operator $\hat{L}^{(0)}$ is Hermitian $\Rightarrow \{W_\alpha^{(0)}\}$ is basis of $\mathcal{H}$
- The solution of the larger tori:

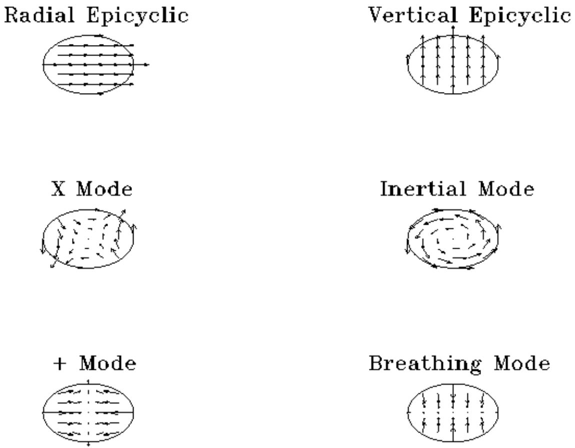
$$W_\alpha = W_\alpha^{(0)} + \beta W_\alpha^{(1)} + \beta^2 W_\alpha^{(2)} + \ldots, \quad W_\alpha^{(n)} = \sum_\gamma c_\gamma^{(n)} W_\gamma^{(0)}$$

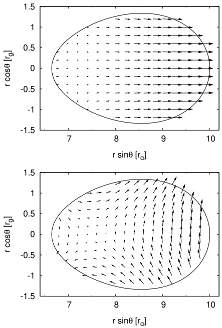
$$\omega_\alpha = \omega_\alpha^{(0)} + \beta \omega_\alpha^{(1)} + \beta^2 \omega_\alpha^{(2)} + \ldots$$

- We finally obtain algebraic equations for $c_\gamma^{(n)}$ in each order of β.

Blaes (1985), Šrámková & Blaes (2007), Straub & Šrámková(2009).

Lowest-order modes of slender and thicker tori





Blaes et al. (2006), Straub & Šrámková(2009).

FEM calculations

Weak form of the problem

FEM use integral formlulation of the problem:

1. By introducing

$$\mathbf{w} \equiv \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad w_1 \equiv W, \quad w_2 \equiv (\omega - m\Omega_1)\, W.$$

   We obtain linear eigenvalue problem:

$$\left[\begin{pmatrix} -m\Omega_1 & -1 \\ \hat{L} & -m\Omega_2 \end{pmatrix} + \omega\right]\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0.$$

2. Define scalar product on $\mathcal{H}$ (2-functions on the torus):

$$\langle \boldsymbol{\varphi} | \mathbf{w} \rangle \equiv \int_{\tilde{S}} \boldsymbol{\varphi}^T \mathbf{w} \mathcal{R} f^{N-1} d\tilde{S}, \quad \boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

   and multiply by *arbitrary* $\boldsymbol{\varphi}$ from left.

Weak form of the problem

We get ($\forall \boldsymbol{\varphi}$):

$$-\int_{\tilde{S}} \tilde{g}^{jk}\left(\tilde{\nabla}_j\varphi_2\right)\left(\tilde{\nabla}_k w_1\right)\mathcal{R}\nu f^N d\tilde{S} - \int_{\tilde{S}} \mathcal{B}\left[\varphi_1 w_2 + m\Omega_1\varphi_1 w_1 + m\Omega_2\varphi_2 w_2\right]\mathcal{R}f^{N-1}d\tilde{S}$$

$$\oint_{\partial\tilde{S}} \varphi_2\left(\tilde{n}^k\nabla_k w_1\right)\mathcal{R}\nu f^N ds + \omega\int_{\tilde{S}} \mathcal{B}\left(\varphi_1 w_1 + \varphi_2 w_2\right)\mathcal{R}f^{N-1}d\tilde{S} \quad = \quad 0.$$

Discretization: *assume* the basis $\{\boldsymbol{\varphi}_a\}$ of $\mathcal{H}$ is finite and expand:

$$\mathbf{w} \approx \mathbf{w}_h \equiv \sum_b W_b\boldsymbol{\varphi}_b = \sum_b W_b\begin{pmatrix}\varphi_{b1}\\ \varphi_{b2}\end{pmatrix}.$$

Taking $\boldsymbol{\varphi} = \boldsymbol{\varphi}_a$ for all a, we get

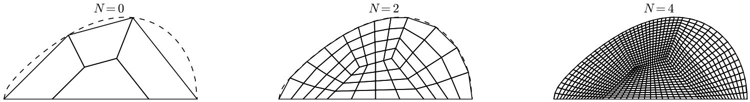
$$\boxed{\sum_b\left(A_{ab} - \omega_h M_{ab}\right)W_b = 0}$$

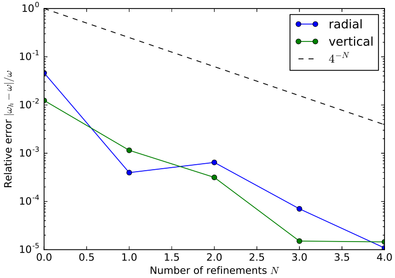
... *Matrix* eigenvalue problem

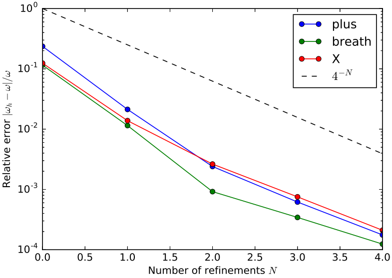
FEM implementation

Diskretized form [$\varphi_b(x)$ = test functions with *limited* support(elements)]:

$$\boxed{\sum_b \left(S_{ab} - \omega M_{ab}\right) W_b = 0,} \qquad W(x) = \sum_b W_b \varphi_b(x)$$



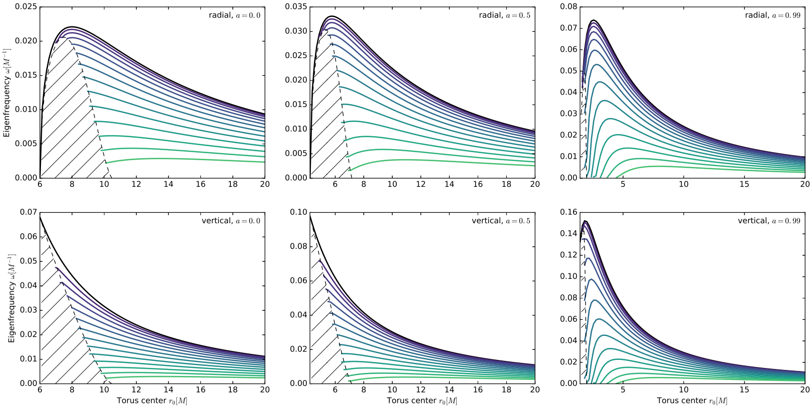




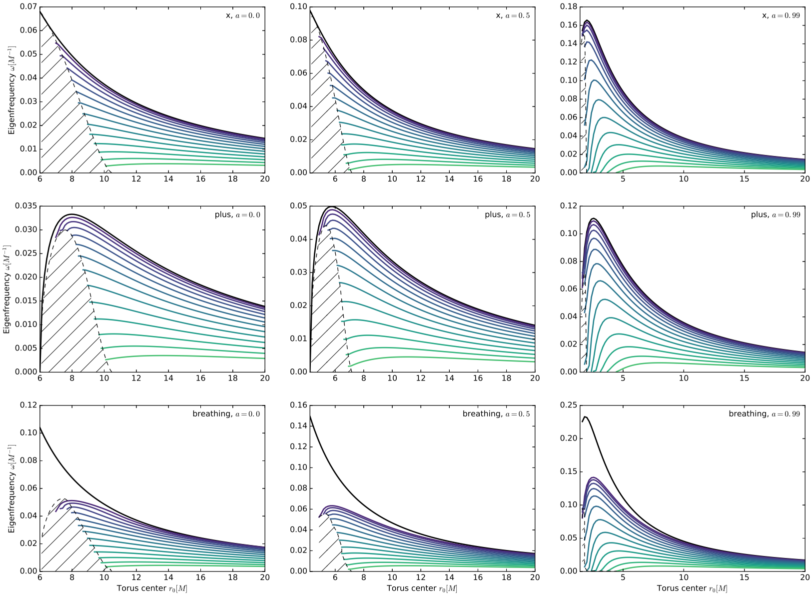
RESULTS

axisymmetric modes

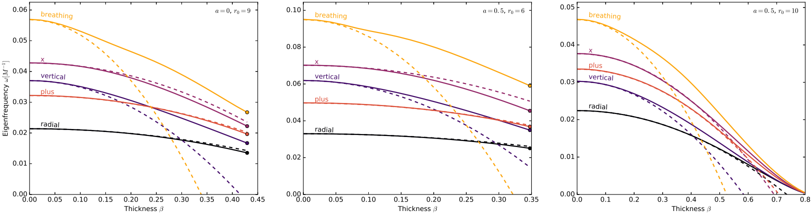
Epicyclic modes



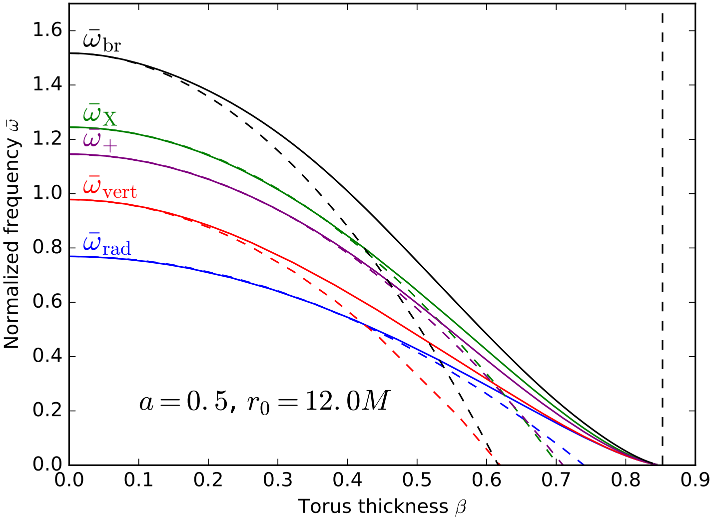
Higher-order modes



Comparison with analytic solution

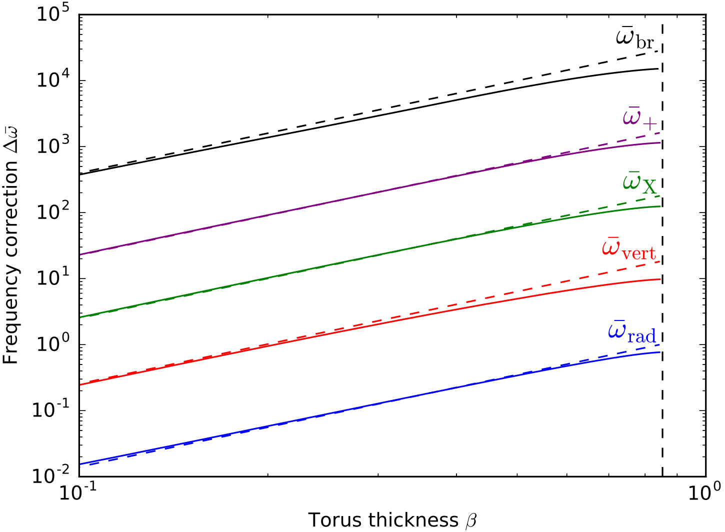


Axisymmetric modes (frequencies)



...Comparison to analytic results of Straub, Šrámková, Blaes

Axisymmetric modes (frequencies)

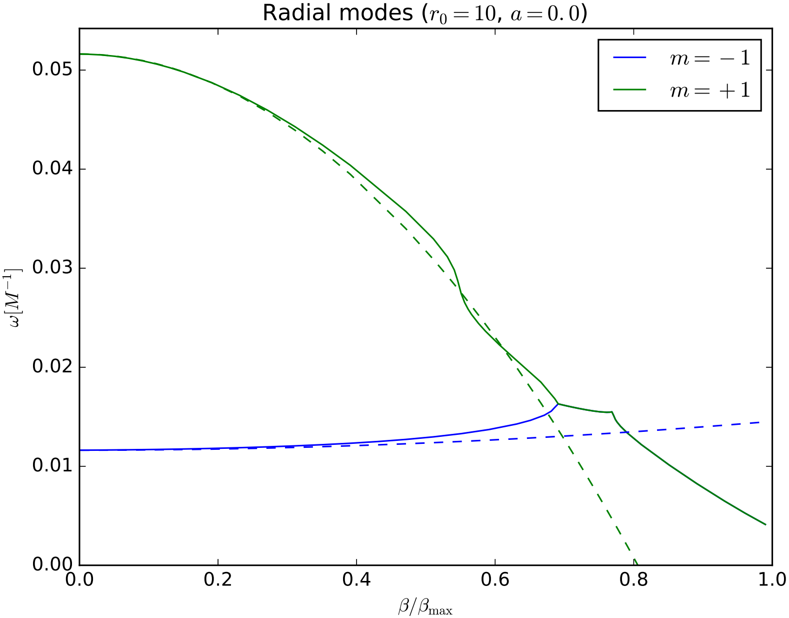


...Comparison to analytic results of Straub, Šrámková, Blaes

RESULTS

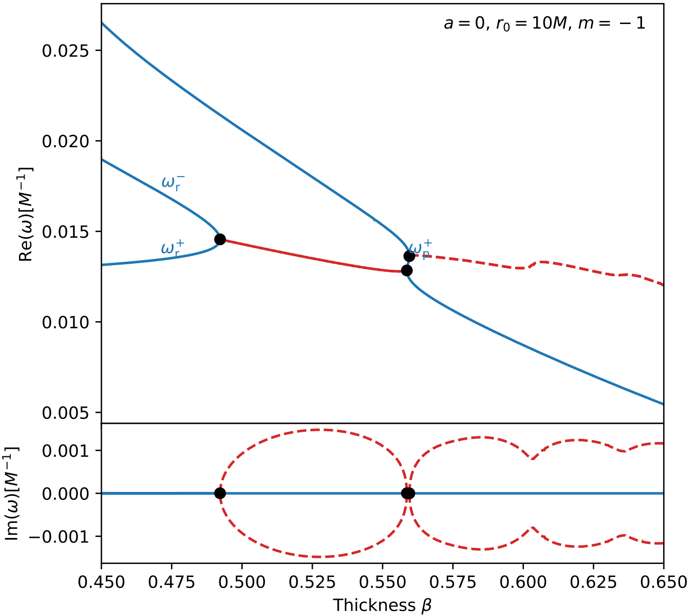
non-axisymmetric modes

Mode merging ($m = 1$ radial modes)

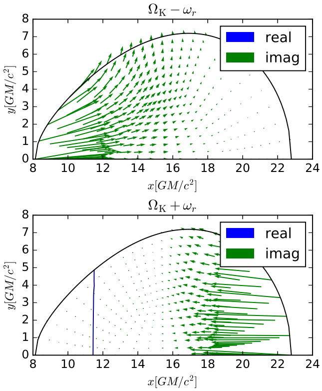


Cusp tori, dashed – analytic predictions of Straub, Šrámková, Blaes

Mode merging



Mode merging causes instability

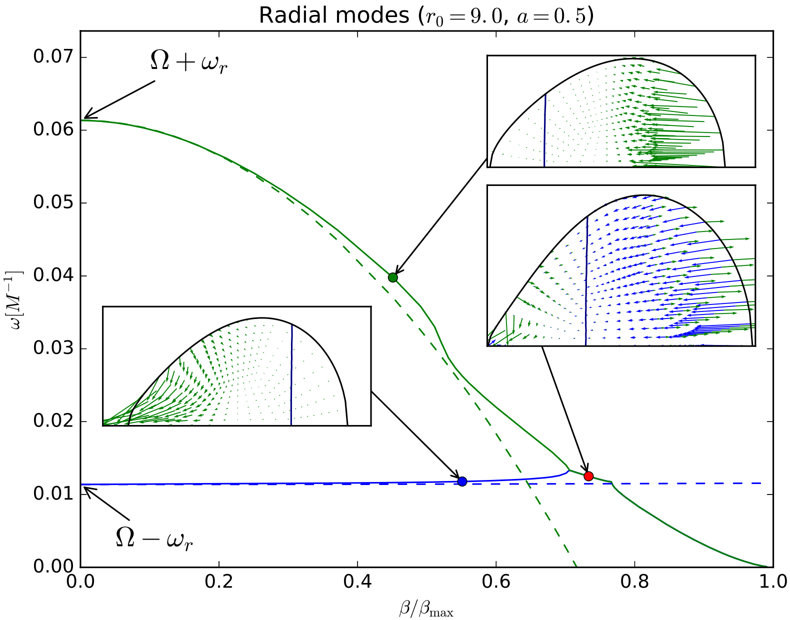


- Inner, $\tilde{\omega} \sim -\omega_r < 0$
- Perturbation **decreases** the total energy of the flow
- **Negative** mode energy

- Outer, $\tilde{\omega} \sim \omega_r > 0$
- Perturbation **increases** flow energy
- **Positive** mode energy

Overreflection $\Rightarrow$ Neutral modes.

Mode merging ($m = 1$ radial modes)



Cusp tori, dashed – analytic predictions of Straub, Šrámková, Blaes

Conclusions

- The analytic results work well for axisymmetric modes.
  → The modes with simple vertical structure are better. (radial, plus and X-mode: $\beta_{\max} \sim 0.5$, vertical and breathing: $\beta_{\max} \sim 0.2$)
- They cannot reveal the instability at finite β.
  → However they well describe PP instability of the corotation mode.
- The well defined modes mix together at higher β.