Dynamical instability of relativistic polytropes in spacetimes with a cosmological constant

RAGtime 22

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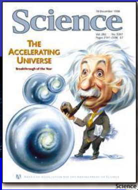


The ‘biggest’ blunder?

The Cosmological Constant Λ has a history as long as GR itself: introduced by Einstein in 1917 to make a static universe, abandoned after Hubble’s discovery of the cosmological expansion, and reintroduced again to explain the accelerated expansion of the Universe (Riess, 1998).

Non-zero Λ?

Observations of the CMB indicate that the ‘dark energy’ must be very close to the *vacuum energy* which can be described by a positive cosmological constant $\Lambda \sim 10^{-52} m^{-2}$ (Ade, 2016).

Type Ia Supernovae and Λ

Observations of SNe Ia with $z \sim 1$, provided evidence that we may live in a low mass-density Universe: $\Omega_M \sim 0.3, \Rightarrow \Omega_\Lambda \sim 0.7$ (Perlmutter, 1999).

Astrophysical implications of Λ

GEODEs

GEneric Objects of Dark Energy (Croker, Nishimura, and Farrah, 2019) are an alternative to black holes, whose interior is described by the 'dark energy' EOS. Examples: de Sitter sphere (Gliner, 1966) and the gravastar (Mazur and Mottola, 2001), formed in the limit of radially decreasing constant-density stars (Mazur and Mottola, 2015; Posada, 2017; Posada and Chirenti, 2019; Chirenti, Posada, and Guedes, 2020).

Relativistic compact objects in the presence of Λ in

- Extension of the interior Schwarzschild solution with Λ (Stuchlík, 2000).
- Λ modifies the Buchdahl bound on the mass-radius ratio M/R (Boehmer, 2004).

The role of Λ in the stability of compact objects

- Radial stability of relativistic fluid spheres using Chandrasekhar's framework (Boehmer and Harko, 2005; Stuchlík and Hledík, 2005).

Polytropes are still relevant!

NS EOS

A long outstanding problem in nuclear physics is determining the correct equation of state (EOS) for cold matter above nuclear saturation density. The only locations in the universe where such matter is believed to exist are the cores of neutron stars (NS). A current goal of relativistic astrophysics is to use NS measurements to constrain the nuclear EOS.

'Piecewise' polytropes

Read et al., 2009, proposed the *piecewise polytropes* (PPs), where the high density region of the NS is approximated by a sequence of polytropic EOS. This approach has been improved by a number of authors (Lindblom, 2010; Alvarez-Castillo and Blaschke, 2017; O'Boyle et al., 2020).

Objective

The purpose of this project was to carry on a detailed study of the role of Λ in the stability of the polytropic fluid spheres, using two different methods, namely, the dynamical approach developed by Chandrasekhar and the critical point method.

Polytropic spheres with Λ

TOV with Λ

- Polytropic fluid spheres (Tooper, 1964),

$$p = K\rho^{1+(1/n)}, \quad n = \text{polytropic index.}$$

- 'Relativistic' parameter $\sigma = (p_c/\rho_c\, c^2)$.

TOV equation with Λ

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -(\epsilon + p)\frac{(G/c^2)m(r) + \left[(4\pi G/c^4)p\frac{\Lambda}{3}\right]r^3}{r^2\left[1 - \frac{2Gm(r)}{c^2 r} - \frac{\Lambda}{3}r^2\right]}; \quad m(r) = 4\pi\int_0^r \rho(r)r^2\,\mathrm{d}r$$

- Radial profiles of the mass density and pressure of the polytropic spheres are given by the relations

$$\rho = \rho_c\theta^n\,, \qquad p = p_c\theta^{n+1}\,.$$

- $\theta(x)$ is a function of the dimensionless radius $x \equiv r/L$,

$$L \equiv \left[\frac{\sigma(n+1)c^2}{4\pi G\rho_c}\right]^{\frac{1}{2}}, \quad \Rightarrow \text{characteristic length scale of the polytropic sphere.}$$

Structure equations with Λ

- To facilitate the numerical computations, it is convenient to introduce dimensionless quantities

$$v(x) \equiv \frac{m(r)}{\mathcal{M}}; \quad \mathcal{M} = 4\pi L^3 \rho_c = \frac{c^2}{G}\sigma L(n+1)$$

- $\lambda \equiv \rho_{\rm vac}/\rho_{\rm c} = \Lambda c^4/8\pi G\rho_{\rm c}$.

Structure equations for polytropes with Λ

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \left[\left(\frac{2\lambda}{3} - \sigma\theta^{n+1}\right)x - \frac{v}{x^2}\right](1+\sigma\theta)\, g_{rr}\,, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = x^2\theta^n,$$

where

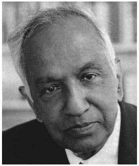
$$g_{rr} \equiv \left[1 - 2\sigma(n+1)\left(\frac{v}{x} + \frac{\lambda}{3}x^2\right)\right]^{-1}.$$

- BCs: $\theta(0) = 1\,, v(0) = 0$.
- The radius $x = x_1$ of the configuration is determined as the first solution of $\theta(x) = 0$.

Dynamical instability via Chandrasekhar's approach

- Condition of dynamical stability in Newtonian theory: $\gamma > 4/3$.
- Relativistic theory of infinitesimal, adiabatic, radial oscillations[†].
- Configuration in hydrostatic equilibrium (TOV equations)

$$\mathrm{d}s^2 = -e^{2\Phi(t,r)}(c\,\mathrm{d}t)^2 + e^{2\Psi(t,r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$



S. Chandrasekhar

- Perturbations preserve the spherical symmetry:

$$q(r,t) = q_0(r) + \delta q(r,t)\,, \quad q \equiv (\Phi, \Psi, \epsilon, p, n).$$

- Stability criterion → normal mode analysis

$$\xi(r,t) = \xi(r)e^{-i\omega t}.$$

- ξ represents the *Lagrangian displacement*, and ω is the frequency of the oscillations.
- $\omega^2 > 0$ (stable); $\omega^2 < 0$ (unstable); $\omega^2 = 0$ (marginally stable).
- Is the Newtonian lower limit $4/3$ on γ affected by GR? Yes, it is ,

$$\gamma_c > \frac{4}{3} + \frac{19}{42}\left(\frac{R_S}{R}\right) + \mathcal{O}\left[\left(\frac{R_S}{R}\right)^2\right], \quad \Rightarrow \quad \text{Constant-density stars.}$$

[†] S. Chandrasekhar (1964). *Astrophys. J.* 140, pp. 417–433

Sturm-Liouville dynamical equation with Λ

Sturm-Liouville dynamic 'pulsation' equation with Λ

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(P\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right)+\left(Q+\omega^2 W\right)\zeta=0\,,\quad \zeta\equiv r^2 e^{-\Phi_0}\xi$$

$$W(r)\equiv\frac{\epsilon_0+p_0}{r^2}\,e^{\Phi_0+3\Psi_0}\,,\quad P(r)\equiv\frac{\gamma p_0}{r^2}\,e^{3\Phi_0+\Psi_0}\,,\quad \gamma=\frac{1}{p\partial N/\partial p}\left[N-(p+\epsilon)\frac{\partial N}{\partial\epsilon}\right],$$

$$Q(r)\equiv\frac{e^{3\Phi_0+\Psi_0}}{r^2}\left[\frac{(p_0')^2}{\epsilon_0+p_0}-\frac{4p_0'}{r}-\left(\frac{8\pi G}{c^4}p_0-\Lambda\right)(\epsilon_0+p_0)e^{2\Psi_0}\right].$$

- BCs: $\xi=0$, at the origin; and $\delta p=0$, at the boundary $r=R$.
- The marginally stable condition $\omega^2=0$ provides the critical adiabatic index γ_{c}. Thus, if $\gamma<\gamma_{\mathrm{cr}}$ dynamical instability will ensue and the configuration will collapse!.
- Λ modifies the relativistic lower limit on γ (Boehmer and Harko, 2005),

$$\gamma_c>\frac{4/3-l}{1-3l}+\frac{19}{42}\left(1-\frac{21}{19}l\right)\left(\frac{R_S}{R}\right)+\mathcal{O}\left[\left(\frac{R_S}{R}\right)^2\right],\quad l\equiv\frac{\Lambda c^4}{12\pi G\epsilon_c}.$$

- Our task: extension of these results for polytropic spheres

Critical adiabatic index for polytropes with Λ

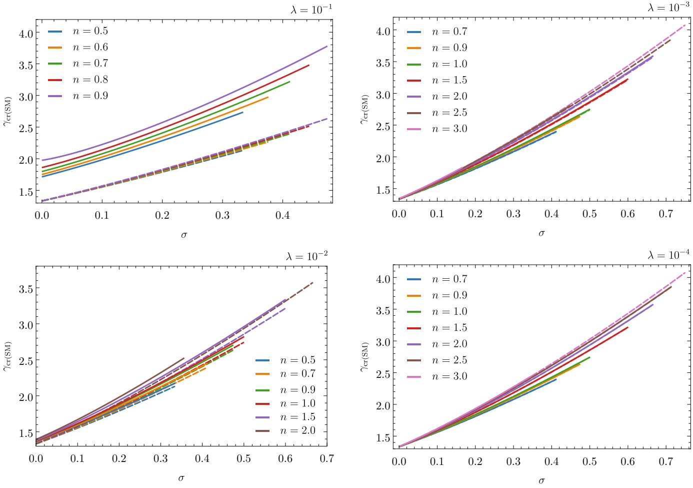
Methods

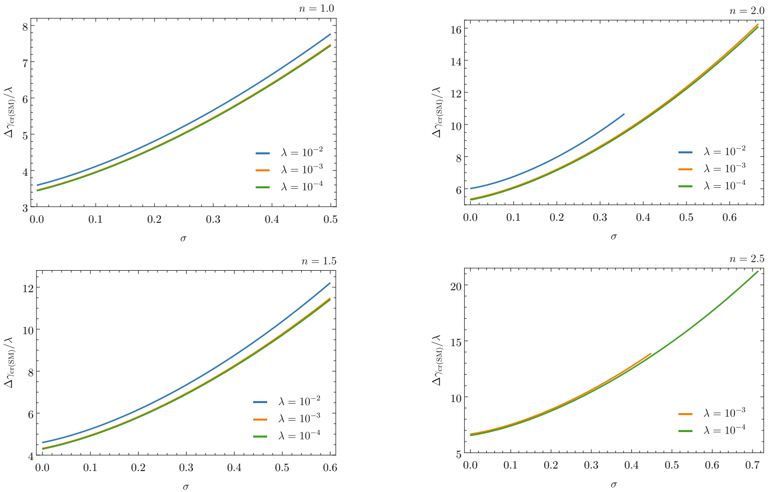
$\ast$ We determined γ_c numerically via two methods:

- Shooting method
- Trial functions

$\ast$ Restriction by the causality limit

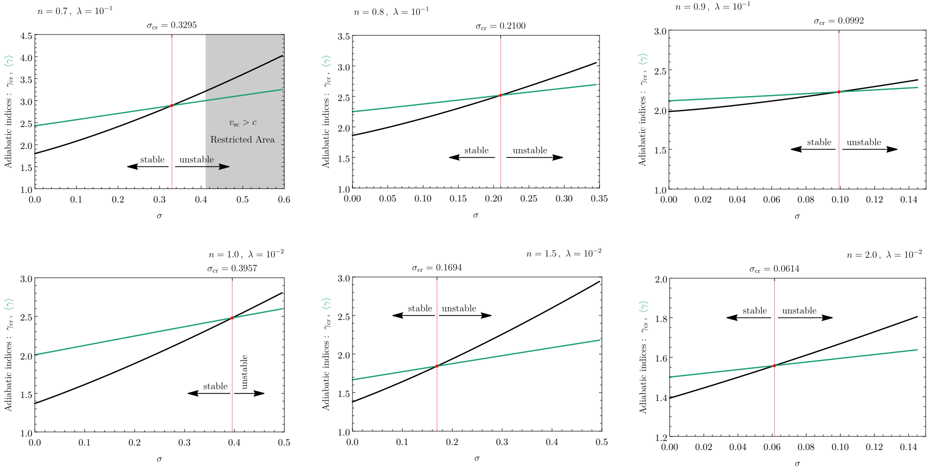
$$\sigma < \sigma_{\text{causal}} = \frac{n}{n+1}$$



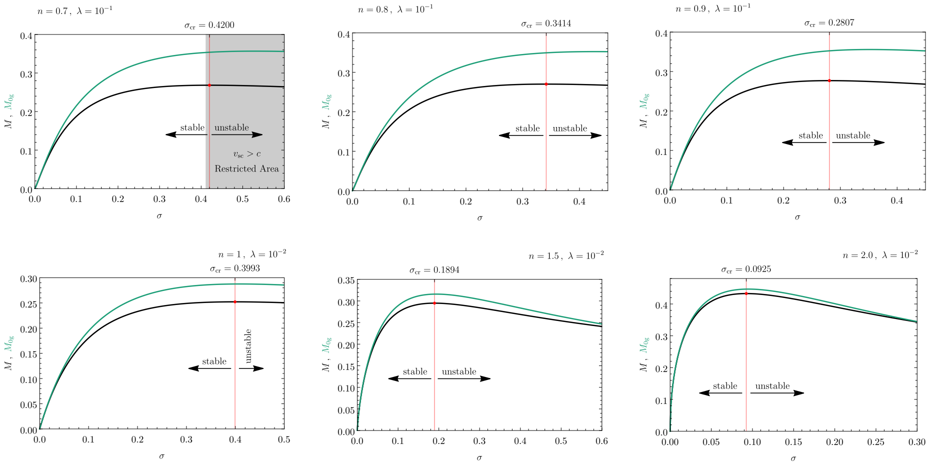


Differences of the critical adiabatic index $\gamma_{\rm cr}$, for the polytropes $n = \{1.0, 1.5, 2.0, 2.5\}$, of the values for $\lambda \in [10^{-6}, 10^{-3}]$ from their corresponding values with $\lambda = 0$.

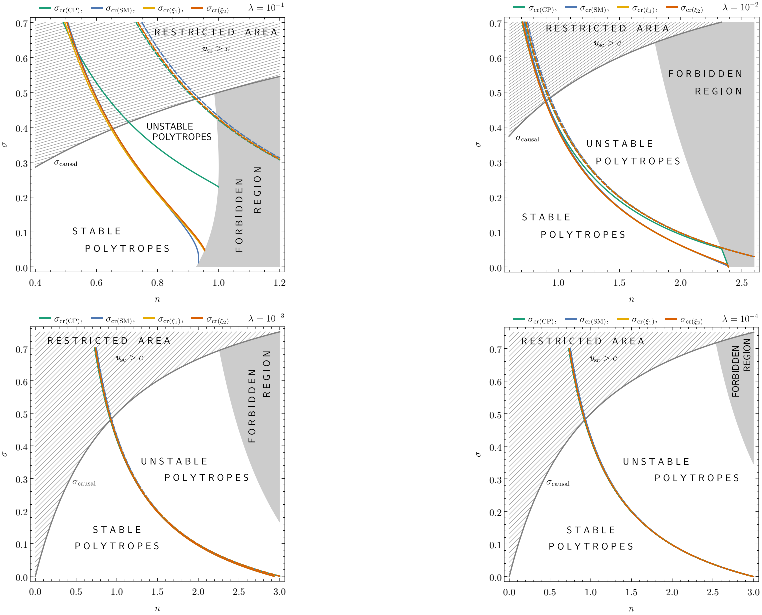
Critical σ determined via the dynamical approach



The stability domain as determined by $\langle\gamma\rangle > \gamma_c$. The values of γ_c were computed via the shooting method.



Total mass (black line) and rest mass (green line), as a function of σ, for some polytropic spheres for different values of λ. The maximum of the curve for the total mass determines the critical value of σ for stability; thus, it separates the stable and unstable regions.



$\sigma_{\mathrm{cr(CP)}}$, $\sigma_{\mathrm{cr(SM)}}$, $\sigma_{\mathrm{cr}(\xi_1)}$, $\sigma_{\mathrm{cr}(\xi_2)}$
$\lambda = 10^{-1}$
RESTRICTED AREA
$v_{\mathrm{sc}} > c$
UNSTABLE POLYTROPES
FORBIDDEN REGION
σ_{causal}
STABLE POLYTROPES
σ
n
$\lambda = 10^{-2}$
RESTRICTED AREA
$v_{\mathrm{sc}} > c$
FORBIDDEN REGION
σ_{causal}
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STABLE POLYTROPES
$\lambda = 10^{-3}$
RESTRICTED AREA
$v_{\mathrm{sc}} > c$
FORBIDDEN REGION
UNSTABLE POLYTROPES
σ_{causal}
STABLE POLYTROPES
$\lambda = 10^{-4}$
RESTRICTED AREA
$v_{\mathrm{sc}} > c$
FORBIDDEN REGION
UNSTABLE POLYTROPES
σ_{causal}
STABLE POLYTROPES

Conclusions

- We found that large values of λ rise the critical adiabatic index γ_c relative to their corresponding values for zero λ. Thus, the cosmological constant tends to *destabilize* the polytropes.
- Our results show that the critical point method and the theory of radial oscillations predict different values of the critical parameter σ_{cr} for nonzero λ.
- Finally, we would like to remark that the role of the vacuum energy on the radial stability of polytropic spheres becomes relevant for the parameter λ sufficiently large, it is negligible for λ smaller than 10^{-4} and becomes significant for λ comparable to 10^{-1}.