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Phase Transitions in Self-Gravitating Systems and Black Holes

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Martin Blaschke, Petr Slaný, FÚ SLU.

Physical reinterpretation of heat capacity discontinuities for static black holes

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(Dated: January 15, 2025)

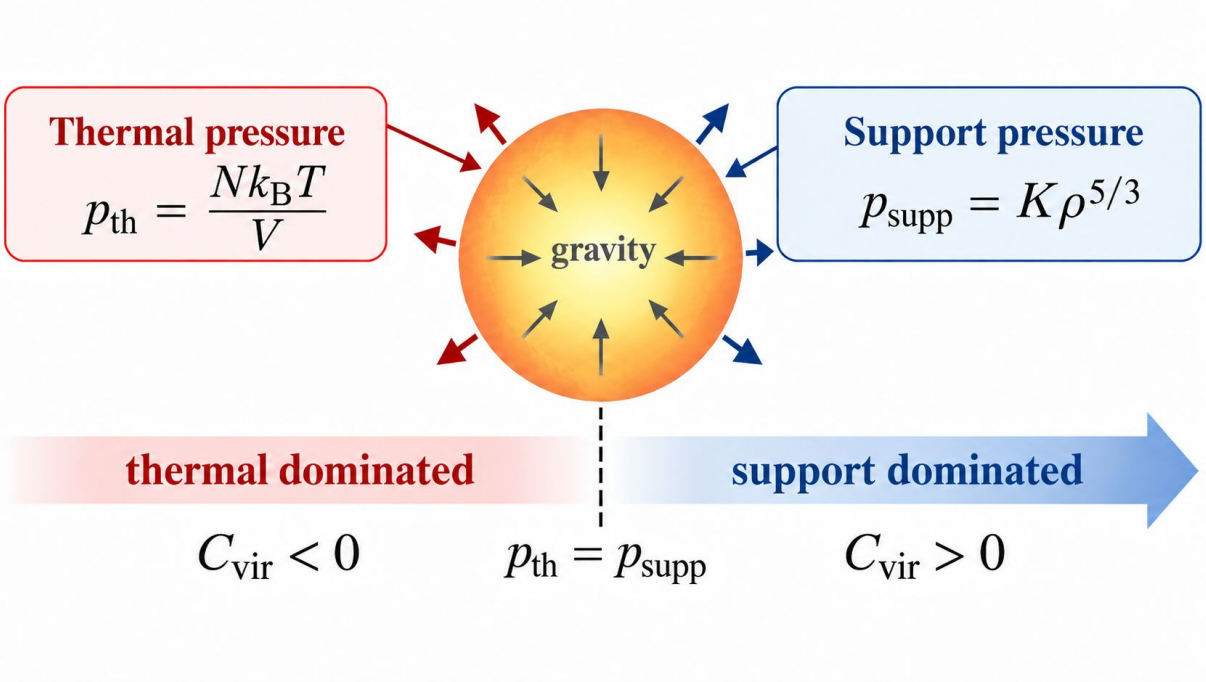
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Caloric Equation of State:

$$U = \frac{3}{2} pV = \frac{3}{2} \left(p_{\mathrm{th}} + p_{\mathrm{supp}} \right) V = \frac{3}{2} NkT + \frac{3}{2} \mathcal{K} \rho^{5/3} V$$

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Generalization to relativistic model:

$$U = \frac{3}{2} NkT + \frac{\mathcal{K}}{n(T) - 1} \rho^{n(T)} V$$

$\mathcal{K}$ is just a constant.

Jüttner gas, Mayer's relation

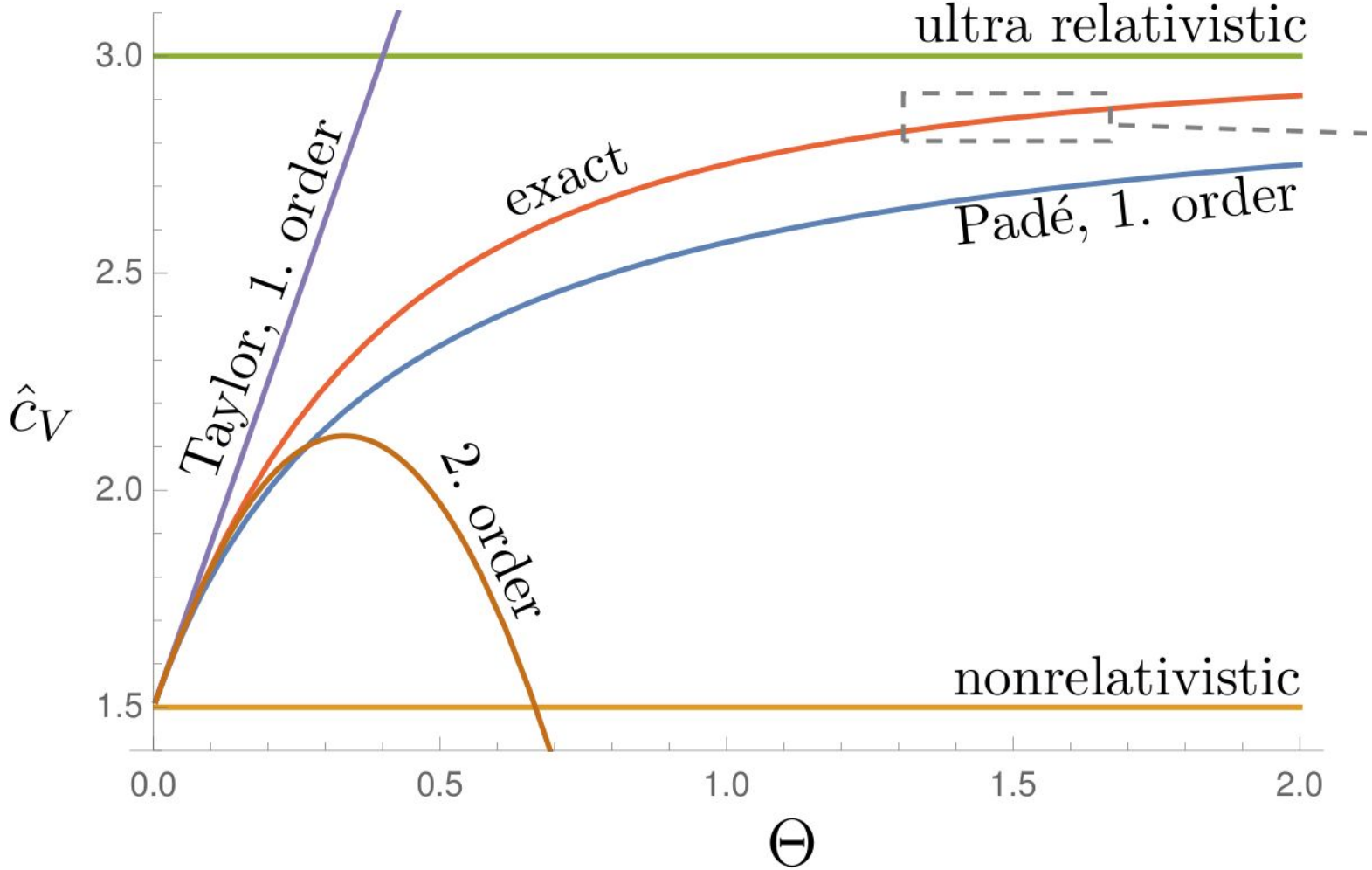
$$n \to n(T) = 1 + \frac{1}{\hat{c}_V(\Theta)}\,, \qquad \Theta = \frac{k_{\mathrm{B}}T}{mc^2}$$

Jüttner gas, Mayer's relation

$$n \to n(T) = 1 + \frac{1}{\hat{c}_V(\Theta)}\,, \quad \Theta = \frac{k_{\mathrm{B}}T}{mc^2}$$

$$\lim_{T\to 0} n(T) = \frac{5}{3}\,, \quad \lim_{T\to\infty} n(T) = \frac{4}{3}$$

$$\lim_{\Theta \to 0} \hat{c}_V(\Theta) = \frac{3}{2}, \quad \lim_{\Theta \to \infty} \hat{c}_V(\Theta) = 3$$


ultra relativistic
exact
Padé, 1. order
Taylor, 1. order
2. order
nonrelativistic
$\hat{c}_V$
Θ
3.0
2.5
2.0
1.5
0.0
0.5
1.0
1.5
2.0

Partition function:

$$z_1 = \frac{4\pi V}{(2\pi\hbar)^3}\left(\int_0^\infty P^2 \mathrm{e}^{-\beta E_k(P)} dP\right), \quad \beta = \frac{1}{k_\mathrm{B} T}$$

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$$E_k = \sqrt{P^2c^2 + m^2c^4} - mc^2$$

$$z_1 = aV\,\mathrm{e}^{1/\Theta}\Theta K_2\left(1/\Theta\right)\ , \quad a = \frac{m^3c^3}{2\pi^2\hbar^3}$$

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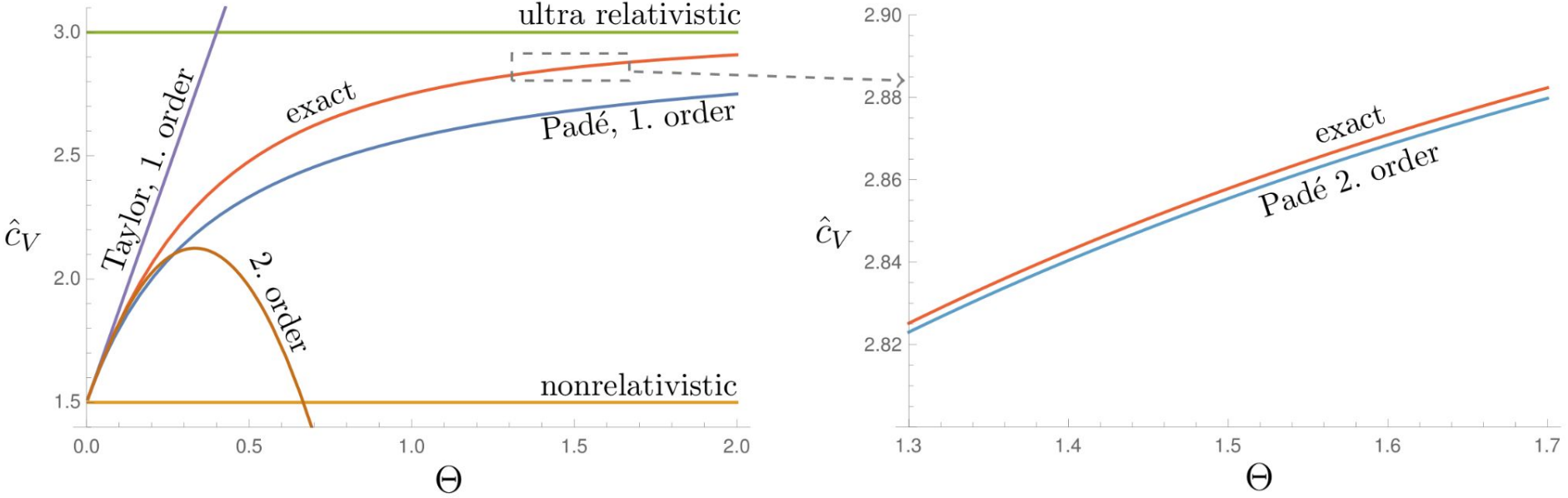
$$F = -k_\mathrm{B}T\ln Z\,, \quad Z = \frac{z_1^N}{N!}$$

$$p = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{Nk_\mathrm{B}T}{V}$$

$$\hat{c}_V = 3 + \frac{1}{\Theta^2} - \frac{3}{\Theta}\frac{K_1\left(1/\Theta\right)}{K_2\left(1/\Theta\right)} - \frac{1}{\Theta^2}\left(\frac{K_1\left(1/\Theta\right)}{K_2\left(1/\Theta\right)}\right)^2$$

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$$\hat{c}_{V\mathrm{P2}} = \frac{3}{2} + \frac{15\Theta + 15\Theta^2}{4 + 10\Theta + 10\Theta^2}$$


ultra relativistic
exact
Padé, 1. order
Taylor, 1. order
2. order
nonrelativistic
$\hat{c}_V$
Θ
3.0
2.5
2.0
1.5
0.0
0.5
1.0
1.5
2.0
2.90
2.88
2.86
2.84
2.82
1.3
1.4
1.5
1.6
1.7

Thermodynamic identity

$$\pi_T \equiv \left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

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Ideal gas:

$$\left(\frac{\partial U_{\mathrm{id}}}{\partial V}\right)_T = 0 \quad \Rightarrow \quad U_{\mathrm{id}} = U_{\mathrm{id}}(T) = \frac{3}{2} N k_{\mathrm{B}} T$$

Thermodynamic identity

$$\pi_T \equiv \left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

Support gas:

$$\left(\frac{\partial U_{\text{supp}}}{\partial V}\right)_T = -\mathcal{K}\rho^{n(T)} \Rightarrow p_{\text{supp}}(T,V) = \mathcal{K}T\int_T^\infty \frac{\rho^{n(\tau)}}{\tau^2}d\tau$$

Virial theorem

$$\Omega + 2U = 0\,, \quad \Omega \text{ is gravitational potential energy}$$

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Relativistic Juttner gas:

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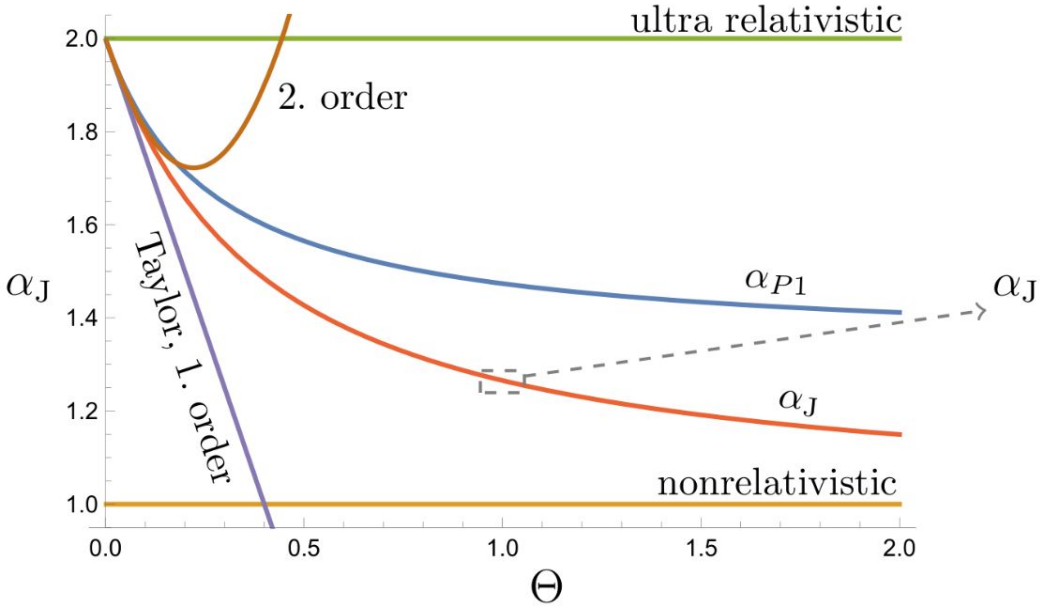
Surrogate model approximation:

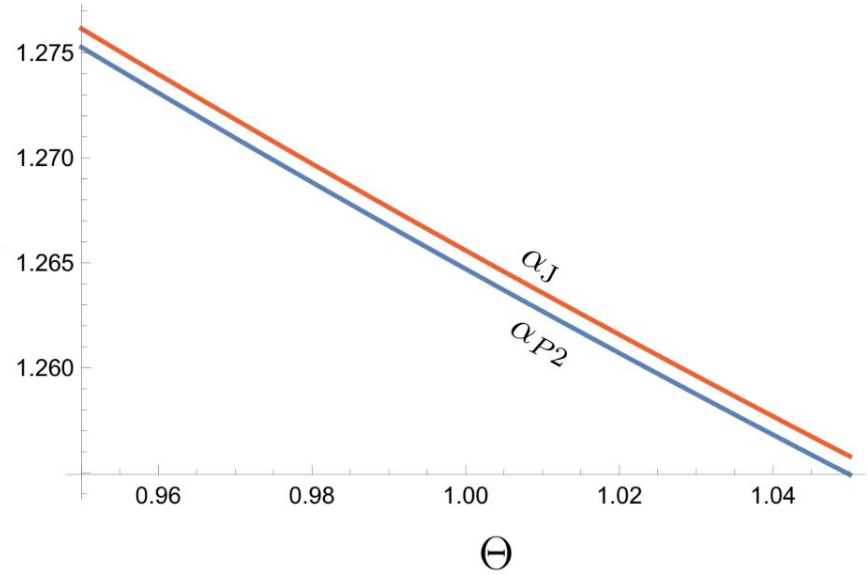
$$\Omega \sim -\alpha_{\mathrm{J}}(\Theta)U$$

$$\alpha_{\mathrm{J}}\left(\Theta\right) = \frac{3\Theta}{\frac{K_3(1/\Theta)}{K_2(1/\Theta)} - \Theta - 1}$$

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$$\alpha_{\mathrm{JP2}}(\Theta) = \frac{8 + 20\Theta + 15\Theta^2}{4 + 15\Theta + 15\Theta^2}$$


ultra relativistic
2. order
Taylor, 1. order
α_{P1}
α_J
nonrelativistic
α_J
Θ
2.0
1.8
1.6
1.4
1.2
1.0
0.0
0.5
1.0
1.5
2.0


α_J
α_{P2}
Θ
1.275
1.270
1.265
1.260
0.96
0.98
1.00
1.02
1.04

$$C_{\mathrm{vir}} = \frac{C_{\mathrm{V}}}{1 - \frac{2T}{p}\left(\frac{\partial p}{\partial T}\right)_V}$$

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$$C_{\mathrm{vir}} \sim \frac{\left(\alpha_{\mathrm{J}}(\theta) - 1\right) C_{\mathrm{V}} + U\,\alpha'_{\mathrm{J}}(\Theta)\frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_V}{\left(\alpha_{\mathrm{J}}(\theta) - 1\right) - \frac{\alpha_{\mathrm{J}}(\Theta)T}{p}\left(\frac{\partial p}{\partial T}\right)_V}$$

Black hole chemistry

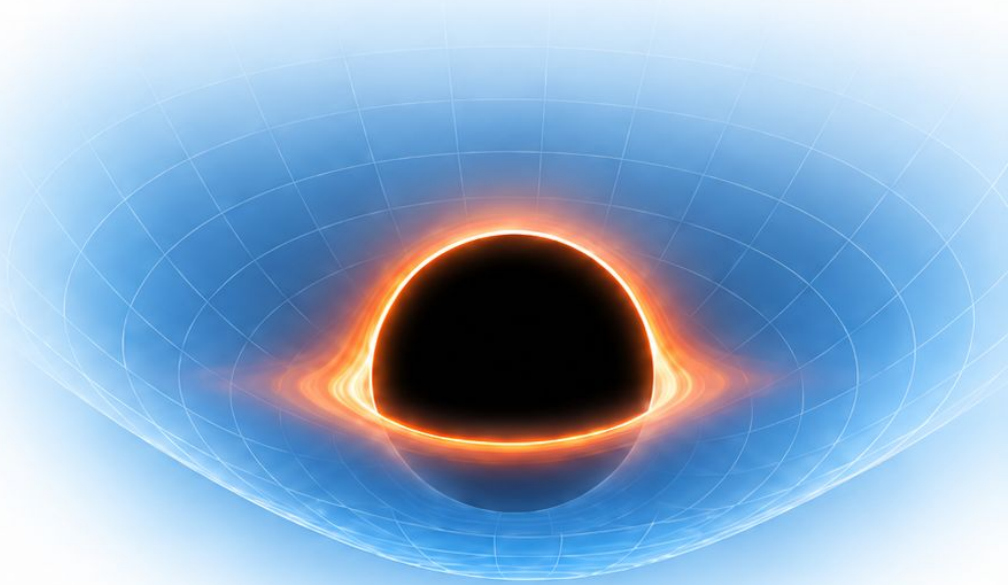
$$p = -\frac{\Lambda}{8\pi}$$

cosmological constant $\longrightarrow$ thermodynamic pressure

$$V = \left(\frac{\partial M}{\partial p}\right)_{S,Q,J}$$

thermodynamic volume

Schwarzschild–AdS: $V = \frac{4}{3}\pi r_+^3$



Kubiznak & Mann, JHEP 1207, 033 (2012)

Schwarzschild AdS Black hole

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2\,, \quad f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}$$

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$$M = \frac{r_+}{2}\left(1 + \frac{8\pi p r_+^2}{3}\right)\,, \quad S = \frac{A}{4} = \pi r_+^2\,, \quad T = \frac{\kappa}{2\pi}$$

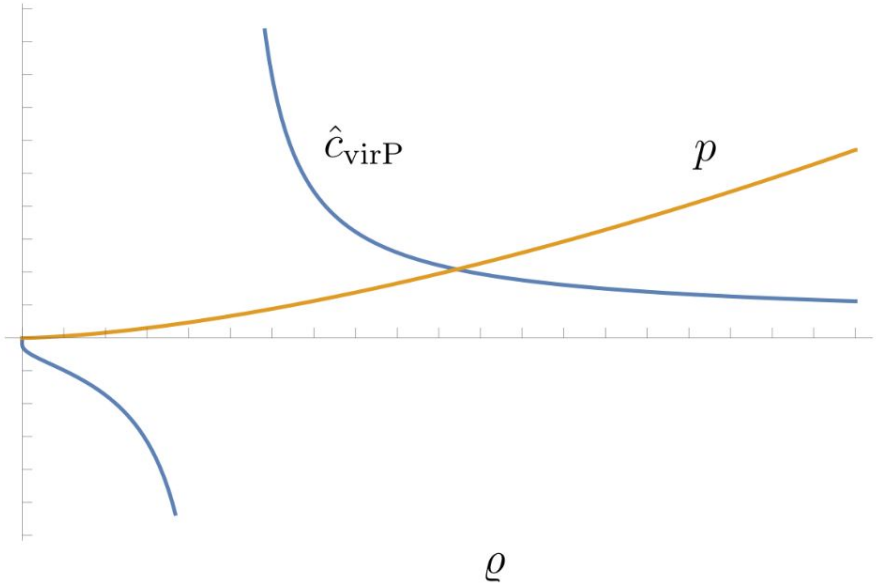
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$$c_p = \left(\frac{\partial M}{\partial T}\right)_p = 2S\frac{1+8pS}{8pS-1}, \quad \rho = \frac{M}{V}$$

Self-gravitating system



Black hole

