Geodesic Chaos

Stationary Case

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Acknowledgments: Prof. O.Semerák (ITP MFF CUNI), P. Suková (ASU CAS), P.Kotlařík(ITP)

Plan of the talk

- Motivation
- Brief summary of the results obtained previously in the static case
- Adding rotatation to the system and studying the dragging effects
- Preliminary results and possible improvements

Motivation

Black Hole accretion system: understanding the motion of test-particles using simplyfied models via numerical simulations which account only for the gravitational fields and adding gradually features that makes the model "more astrophisical" as possible.

GR and chaos

General relativity is an highly non linear theory

⇒ **relativistic system prone to chaos**:

- Cosmological model
- test-particle motion around black hole ✓

Kerr-Newman family: isolated, stationary, asymptotically flat BH solution are full integrable but...

⇒ **small perturbation lead to chaotic dynamics**:

- spin of the test-particle
- background perturbations ✓

⇒ **there are several type of background perturbations**:

- additional multipoles
- magnetic fields
- adding exact fields with specific symmetries ✓

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Integrability of geodesic motion

1. Schwarzschild: static, spherically symmetric space-time, asymptotically flat (vacuum)
    - **Energy** ⇔ Stationarity (R) time-traslation(1 killing vector)
    - $L = (L_x, L_y, L_z)$ ⇔ Spherical symmetry SO(3) (3 killing vector)
    - **Particle's mass** ⇔ normalization of four-velocity

2. Kerr: stationary, axi-symmetric space-time, asymptotically flat (vacuum)
    - **Energy** ⇔ Stationarity (R), time-translation (1 killing vector)
    - L_{axis} ⇔ SO(2)(1 killing vector)
    - **Particle's mass** ⇔ normalization of four-velocity
    - **Carter constant** ⇔ corresponds to the total angular momentum plus a precisely defined part which is quadratic in the linear momenta (1 killing tensor)

Why Geodesic Chaos?

Realistic astrophysical model of black holes are non-isolated and the surrounding material may affect higher derivatives of the metric leading to a destabilization of the motion and, in general, to the occurrence of chaos.

Assumptions

1. Stationarity
2. Matter flow has axially symmetric disc geometry and follows circular orbits (circular spacetimes)

Physical Interpretation of the discs

- one-component perfect fluid with proper azimuthal pressure, proper surface density and velocity
- two-component dust on circular orbits about the central black hole which in the stationary case are two counter-rotating geodesic. streams characterized by their proper densities $\sigma\pm$ and velocities $v\pm$.

Weyl metric in cylindrical coordinates

$$\boxed{ds^2 = -e^{2\nu(\rho,z)}dt^2 + \rho^2 e^{-2\nu(\rho,z)}d\phi^2 + e^{[2\lambda(\rho,z)-2\nu(\rho,z)]}(d\rho^2 + dz^2)}$$

$$\nabla^2\nu = 0 \qquad \lambda = \int_{axis}^{\rho,z} \rho\left[(\nu_{,\rho}^2 - \nu_{,z}^2)d\rho + 2\nu_{,\rho}\nu_{,z}\, dz\right]$$

Constants of motion

- **energy**: $E = e^{2\nu} u^t$
- **azimuthal angular momentum**: $L_z = \rho^2 e^{-2\nu} u^\phi$
- **mass of the particle**: $g_{\mu\nu} u^\mu u^\nu = -1$

No complete integrability of the motion (in general)

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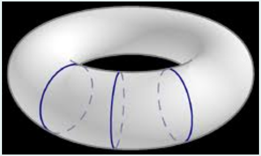
Reduced Lagrangian

$$-e^{-2\nu}E^2 + e^{2\nu}\frac{L_z^2}{\rho^2} + e^{2(\lambda-\nu)}\left[(u^\rho)^2 + (u^z)^2\right] = -1$$

The non trivial motion is characterized by (ρ, z, u^ρ, u^z). It means that it is equivalent to a 2d Hamiltonian system and it is confined on 3d hypersurfaces.



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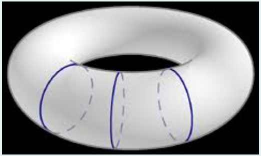

Poincaré surfaces of section

Let be H an Hamiltonian autonomous system with $2n$ degree of freedoms. Since the energy is conserved the phase space can be reduced to $2n-1$-surfaces.
A surface of section is then obtained by

1. $q_i = const$, set another degree of freedom as constant
2. take the value of the other $2n-2$ degrees of freedom $(p_1, \dots, p_{(n-2)}, q_1, \dots, q_{(n-2)})$, each time the orbits cross the hyper surface defined by $q_i = const$ (in a fixed direction)

Remarks

- Resonant tori: manifest itself as infinite set of points;
- Non- resonant tori: appear as a succession of points which cover densely the invariant curves.
  PS give an overall view of the dynamics of the system and of the accessible states of the system

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Free motion around black hole with discs or rings: between integrability and chaos- I-II-III

Perturbation scheme: $\nu = \nu_{Schw} + \hat{\nu}$

- Bach Weyl ring
- inverted 1st/4th counter-rotating Morgan-Morgan disc
- power law disc

Methods for detecting chaos:

- Poincaré surfaces of section for $z = 0$
- time series and the corresponding power spectra of $(u^r, z,$ **longitudinal action**$)$
- recurrence plots
- Kaplan Glass method based on tracing directions
- Lyapunov-type coefficients (mLCE, FLI, MEGNO)

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Summary of results for Poincaré surfaces of section:

1. The dynamics tends to be regular for both very small and very large values of (E, L_z) and the parameters of the disc $(\mathcal{M}, r_{disc})$;
2. The more compact is the source, the more chaotic is the behaviour.

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References for the static setting

1. O. Semerák, P. Suková, Free motion around black holes with discs or rings: between integrability and chaos – I, Monthly Notices of the Royal Astronomical Society, Volume 404, Issue 2, May 2010, Pages 545–574,
2. Semerák, O., Suková, P. (2010). Free motion around black holes with discs or rings: between integrability and chaos - II: Chaos around black holes with discs or rings. Monthly Notices of the Royal Astronomical Society, 404, 2455-2476.
3. Suková Petra , Semerak Oldrich. (2013). Free motion around black holes with discs or rings: Between integrability and chaos-III. Monthly Notices of the Royal Astronomical Society.
4. V. Witzany, O. Semerák and P. Suková, "Free motion around black holes with discs or rings: between integrability and chaos – IV," Mon. Not. Roy. Astron. Soc. **451** (2015) no.2, 1770-1794
5. L. Polcar, P. Suková and O. Semerák, "Free Motion around Black Holes with Disks or Rings: Between Integrability and Chaos–V," Astrophys. J. **877** (2019) no.1, 16
6. L. Polcar and O. Semerák, "Free motion around black holes with discs or rings: Between integrability and chaos. VI. The Melnikov method," Phys. Rev. D **100** (2019) no.10, 103013

Bardeen-Torne-Cartan metric in isotropic coordinates

$$ds^2 = -e^{2\nu(r,\theta)}dt^2 + B(r,\theta)^2 r^2 e^{-2\nu(r,\theta)} \sin^2\theta (d\phi - \omega(r,\theta)dt)^2 + e^{[2\lambda(r,\theta)-2\nu(r,\theta)]}(dr^2 + r^2 d\theta^2)$$

Remarks

- **Weyl coordinate** $\rho = r\sin\theta, \quad z = r\cos\theta$
- ν, λ, ω, B to be determined by Einstein field equations

B: $$B_{,\rho\rho} + \frac{2B_\rho}{\rho} + B_{,zz} = 8\pi B(T_{\rho\rho} + T_{zz})$$

vacuum : $$T_{\mu\nu} = 0 \quad \begin{cases} B = 1 \\ B = 1 - \frac{M^2}{4(\rho^2+z^2)} = 1 - \frac{M^2}{4r^2} \end{cases} \Rightarrow \textbf{horizon} \begin{cases} \rho = 0, \quad |z| \le M \\ r = \frac{M}{2} \end{cases}$$

- After applying adequate boundary conditions at infinity, on the axis and at the horizon, the remaining non-liner coupled equations must be solved for ν and ω and finally λ is obtained by line integration.
- Analytically solution only for static case $\omega = 0$
- Non static case:
  1. generating technique
  2. perturbative approach ✓

Slowly rotating thin disc with constant Newtonian surface density

- **Will 1974:** Schwarzschild BH plus a slowly rotating light concentric thin ring obtained in terms of a multipole expansion of the mass and spin perturbation series.
- **P. Čížek and O.- Semerák 2017:** at the first perturbative order obtained the solution, in terms of elliptical integrals, of a rotating disc with a constant Newtonian surface density encircling a Schwarzscild BH.
  - B can be chosen as above mentioned
  - ν will be the sum of the Schwarzschild part plus the disc potential:

$$\nu(x,\theta) = \nu_{Schw} + V(x' = x_{out},\theta) - V(x' = x_{out},\theta)$$

$$V(x';x,\theta) = 2\Pi S|\tilde{z}|H(\tilde{\rho}' - \tilde{\rho}) - \frac{2S}{\sqrt{(\tilde{\rho}' + \tilde{\rho})^2 + \tilde{z}^2}} \Big\{ \left[(\tilde{\rho}' + \tilde{\rho})^2 + \tilde{z}^2\right] E(k) \quad (1)$$

$$+ (\tilde{\rho}'^2 - \tilde{\rho}^2)K(k) + \tilde{z}^2 \frac{\tilde{\rho}' - \tilde{\rho}}{\tilde{\rho}' + \tilde{\rho}} \Pi \left[\frac{4\tilde{\rho}'\tilde{\rho}}{(\tilde{\rho}' + \tilde{\rho})^2}, k\right] \Big\}$$

  - ω is a very long expression that can be found in the references
  - λ can be obtained as line integral as in the static case (at first order)

Asymptotic behaviour: $r \to \infty$

$$\nu \propto -\frac{M + \mathcal{M}}{r}, \quad \omega \propto \frac{2\mathcal{J}}{r^3}, \quad \lambda \propto -\frac{M^2}{4r^2} \quad (2)$$

Reduced Lagrangian stationary case

$$-e^{-2\nu}(E - L_z\omega)^2 + e^{2\nu}\frac{L_z^2}{\rho^2} + e^{2(\lambda-\nu)}\left[(u^\rho)^2 + (u^z)^2\right] = -1$$

Reduced Lagrangian static case

$$-e^{-2\nu}E^2 + e^{2\nu}\frac{L_z^2}{\rho^2} + e^{2(\lambda-\nu)}\left[(u^\rho)^2 + (u^z)^2\right] = -1$$

The allowed region of the phase-space changes

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References for this perturbative solution

1. Semerák, O.; Čížek, P. Rotating Disc around a Schwarzschild Black Hole. Universe 2020, 6, 27.
2. P. Čížek and O. Semerák 2017 ApJS 232 14
3. P. Kotlařík, O. Semerák and P. Čížek, "Schwarzschild black hole encircled by a rotating thin disc: Properties of perturbative solution," Phys. Rev. D **97** (2018) no.8, 084006

GRAVIT: the implementation of the solution and the numerical study

GRAVIT,1988,Miroslav Zacek

The code used to study the geodesic motion in a given spacetime. It is written in c++ and adjusted by P. Suková for the static study. The algorithm used for the integration of the motion is the Huta RK of 6th order with 8 correctors.

GRAVIT II, 2021-22

It is the code modified by me to implement the above solution and verified the hypothesis that the dragging effect would led to a damping of the chaotic behaviour of the orbits.

$x_{out} = 6 \; x_{in} = 5, \; S = 0.001, \; E = 0.995 \; L_z = 3.75, \; \tau = 100000M$

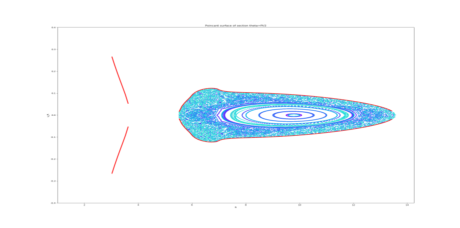


Figure: W=0.0

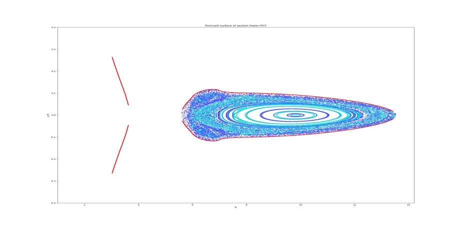


Figure: W=0.10

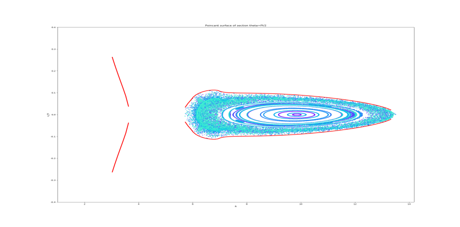


Figure: W=0.20

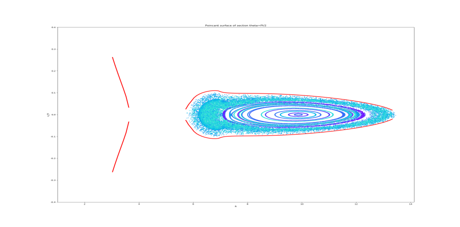
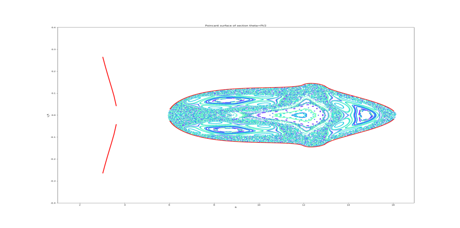


Figure: W=0.25

$x_{out} = 12$ $x_{in} = 11$, $S = 0.001$, $E = 0.995$ $L_z = 3.75$, $\tau = 100000M$



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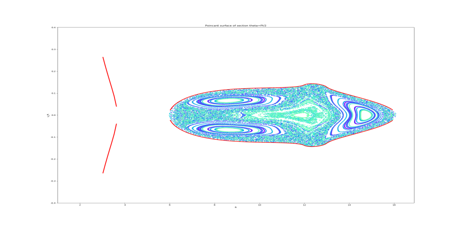


Figure: W=0.2

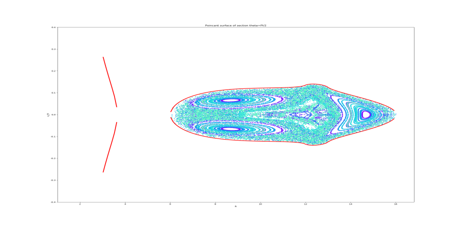


Figure: W=0.6

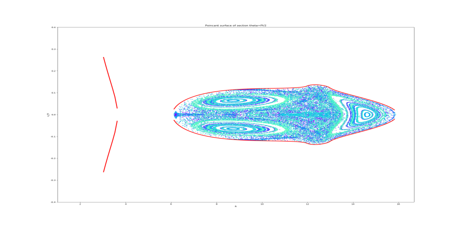


Figure: W=1.0

$x_{out} = 30 \; x_{in} = 29, \; S = 0.001, \; E = 0.995 \; L_z = 3.75, \; \tau = 100000M$

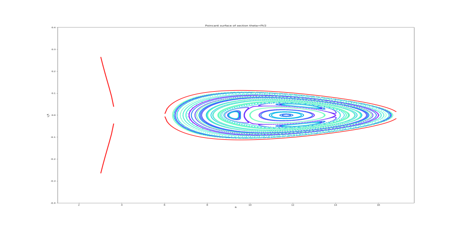


Figure: W=0.0

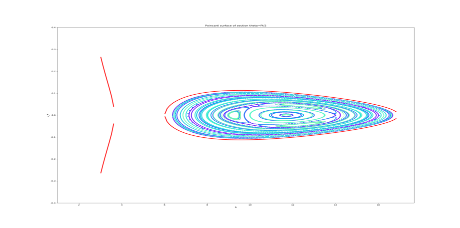


Figure: W=0.10

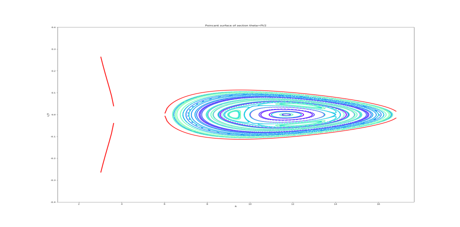


Figure: W=0.20

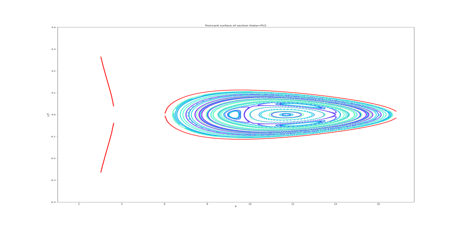
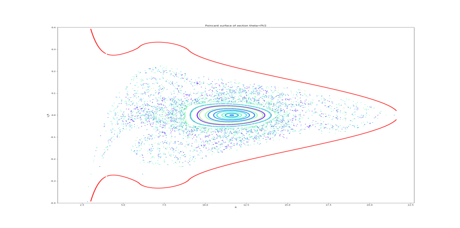


Figure: W=0.25

$x_{out} = 8$ $x_{in} = 5$, $S = 0.002$, $E = 0.995$ $L_z = 3.75$, $\tau = 100000M$



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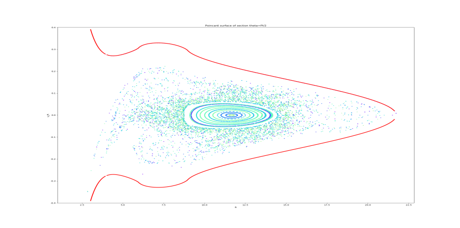

Figure: W=0.10

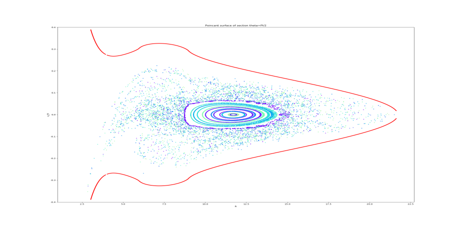

Figure: W=0.20

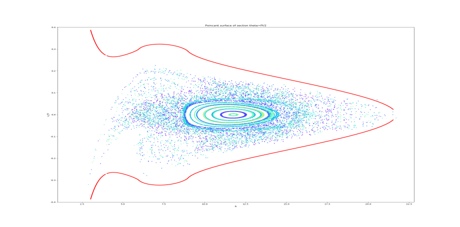

Figure: W=0.30

Rotation and chaos in the literature

1. **Chaos and rotating black holes with halos**
   P.S. Letelier (Campinas State U., IMECC), W.M. Vieira (Campinas State U., IMECC), Phys.Rev.D 56 (1997), 8095-8098
2. **Stability of Orbits around a Spinning Body in a Pseudo-Newtonian Hill Problem**
   A.F. Steklain (Campinas State U., IMECC), P.S. Letelier (Campinas State U., IMECC), Phys.Lett.A 373 (2009), 188-194
3. **Influence of the black hole spin on the chaotic particle dynamics within a dipolar halo**
   Sankhasubhra Nag, Siddhartha Sinha, Deepika B. Ananda, Tapas K Das, Astrophys.Space Sci. 362 (2017) 4, 81
4. **Stealth Chaos due to Frame Dragging**
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Overall picture

- The frame dragging induced by rotation of the system seems to lead to a suppression of the chaotic behaviour
- The counter-rotating motion appears to be more unstable than the co-rotating motion

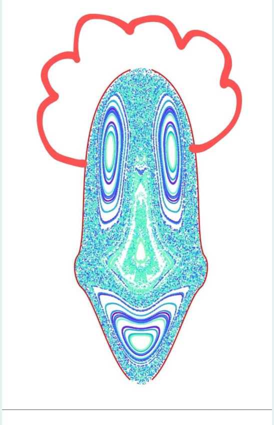
Preliminary results

Increasing the angular momentum of the disc, apparently, the chaotic behaviour of the dynamics seems to decrease.

- **Improvements**

$\Rightarrow$ The accuracy of the integrator can be improved implementing in the code another integrator with adaptive step-size;

$\Rightarrow$ Turn back to the static case and analyse the effect of the edge/s on the chaotic behaviour comparing the first order Morgan-Morgan inverted disc, with an edge, with the family of the Inverted inverted Kuzmin-Toomre discs, which have no edges.



Thank you!