Can extended bodies follow geodesic orbits?

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Motivation

- "Precisely computing bound orbits of spinning bodies around black holes I and II"(arxiv:2201.13334,2201.13335) uses:
  - $P^\mu = m\mathcal{U}^\mu$ assumption.
  - a two worldline approach (geodesic + spin corrected).
- Remaining strictly in the Mathisson "skeleton" framework, can
  - $P^\mu = m\mathcal{U}^\mu$?
  - an extended body follow a geodesic orbit?

pole-dipole-quadrupole body

Mathisson-Papapetrou-Dixon (MPD) equations of a body

$$\begin{aligned} \dot{P}^{\mu} &= -\frac{1}{2} R^{\mu}{}_{\nu\alpha\beta} \mathcal{U}^{\nu} S^{\alpha\beta} - \frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla^{\mu} R_{\alpha\beta\gamma\delta}, \\ \dot{S}^{\mu\nu} &= 2 P^{[\mu} \mathcal{U}^{\nu]} + \frac{4}{3} J^{\alpha\beta\gamma[\mu} R^{\nu]}{}_{\gamma\alpha\beta}, \end{aligned}$$

To fix the center of the mass we need a Spin Supplementary Condition (SSC):

$$S^{\mu\nu} w_{\mu} = 0, \;\; w^{\mu} w_{\mu} = -1.$$

w^{μ} is just a future pointing unit vector.

The Ohashi-Kyrian-Semerák (OKS) SSC

$$P^\mu = P^\mu_\parallel + P^\mu_{\mathrm{hid}},$$

$$P^\mu_\parallel = m\mathcal{U}^\mu$$

$$P^\mu_{\mathrm{hid}} = (\delta^\mu_{\ \nu} + \mathcal{U}^\mu \mathcal{U}_\nu) P^\nu$$

Choosing $\dot{w}^\mu = 0$ leads to $w_\mu \dot{S}^{\mu\nu} = 0$ and

$$P^\mu = \frac{1}{-w_\nu \mathcal{U}^\nu}[(-P^\gamma w_\gamma)\mathcal{U}^\mu + \frac{4}{3} J^{\alpha\beta\gamma[\mu} R^{\delta]}_{\ \ \gamma\alpha\beta} w_\delta].$$

For pole-dipole $P^\mu_{\mathrm{hid}} = 0$ (Kyrian & Semerák, MNRAS (2007)).
For pole-dipole-quadrupole can $J^{\alpha\beta\gamma[\mu} R^{\delta]}_{\ \ \gamma\alpha\beta} w_\delta = 0$?

OKS going quadrupole

Assume that there is such a w^μ that $P^\mu = m\mathcal{U}^\mu$, then

$$\dot{P}^\mu = \dot{m}\mathcal{U}^\mu + m\dot{\mathcal{U}}^\mu$$

and from MPD

$$\dot{m} = \frac{1}{6} J^{\alpha\beta\gamma\delta}\mathcal{U}_\mu \nabla^\mu R_{\alpha\beta\gamma\delta}.$$

There is no guarantee even when $P^\mu || \mathcal{U}^\mu$, that their derivatives will remain parallel.

$$\dot{S}^{\mu\nu} = \frac{4}{3} J^{\alpha\beta\gamma[\mu} R^{\nu]}{}_{\gamma\alpha\beta} := K^{\mu\nu}$$

implies that $\dot{K}^{\mu\nu} w_\mu = 0$, $\ddot{K}^{\mu\nu} w_\mu = 0$,...

Pole-dipole bodies on geodesics

Schwarzschild

- Only geodesic radial motion is compatible with the MPD under OKS SSC in the pole-dipole approximation.
- The existence was known (Costa and Natário, Fund. Theor. Phys. (2015)), but not the uniqueness.

Kerr equatorial

- Only geodesics obeying $J_z = aE$ are compatible with the MPD under OKS SSC in the pole-dipole approximation.
- Existence and uniqueness provided for the first time.

In these cases OKS SSC centroid coincides with the Mathisson-Pirani SSC $S^{\mu\nu}\mathcal{U}_\nu = 0$ and the Tulzcyjew-Dixon $S^{\mu\nu}P_\nu = 0$.

Pole-dipole-(spin induced)quadrupole body on geodesics?

- Used spin induced quadrupole

$$J^{\alpha\beta\gamma\delta} = -3w^{[\alpha}Q^{\beta][\gamma}w^{\delta]}, \text{where} \quad Q^{\beta\gamma} = C_{S^2}S^{\beta}{}_{\alpha}S^{\alpha\gamma}.$$

- Imposed geodesic trajectories compatible with the pole-dipole cases in Schwarzschild and Kerr.
- Chosen w^{μ} such that $J^{\alpha\beta\gamma[\mu}R^{\delta]}{}_{\gamma\alpha\beta}w_{\delta} = 0$ implying $P^{\mu}_{\mathrm{hid}} = 0$.

We have shown that the above setup is not compatible with a pole-dipole-(spin induced)quadrupole body.

Bending the rules

- Assume there is no interaction between the dipole and the quadrupole terms; the body has only mass quadrupole and it is moving on Schwarzschild background.
- From the pole-dipole we know that the body can move on a radial geodesic. This leads to $\dot{S}^{\mu\nu} = K^{\mu\nu} = 0$, which implies that $K^{\mu\nu} w_\nu = 0 \; \forall \; w^\nu$, i.e. the hidden momentum vanishes.
- The vanishing spin quadrupole coupling leads to $\dot{m} \, \mathcal{U}^\mu = -\frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla^\mu R_{\alpha\beta\gamma\delta}$.
- All the above boils down to ...

...couplings between $Q^{\mu\nu}$ and w^{μ}.

$$
\begin{aligned}
Q^{tr} &= \frac{1}{2w^t w^r}(Q^{rr}[w^t]^2 + Q^{tt}[w^r]^2) \\
&+ \frac{rQ^{\theta\theta}}{2(r-2M)w^t w^r}\Big\{ -(r-2M)^2[w^t]^2 + r^2[w^r]^2 \Big\}, \\
Q^{t\theta} &= \frac{Q^{r\theta}w^t}{w^r}, \quad Q^{\theta\phi} = \frac{Q^{r\theta}w^\phi}{w^r}, \\
Q^{t\phi} &= \frac{Q^{r\phi}w^t}{w^r} + \frac{w^\phi}{2w^t[w^r]^2}\Big\{ Q^{tt}[w^r]^2 \\
&- Q^{rr}[w^t]^2 - \frac{rQ^{\theta\theta}[r^2(w^r)^2 + (2M-r)^2(w^t)^2]}{2M-r} \Big\}, \\
Q^{\phi\phi} &= \frac{1}{[w^r]^2}\Big[Q^{\theta\theta}([w^r]^2 + (r-2M)[w^\phi]^2) + w^\phi(2Q^{r\phi}w^r - Q^{rr}w^\phi) \Big].
\end{aligned}
$$

The MPD do not evolve the quadrupole tensor, but the above constrains dictate the way the mass quadrupole tensor evolves under OKS SSC.

Similarly as in Schwarzschild, but for $J_z = aE$ trajectories in Kerr, using the Carter tedrad field we end up with

$$Q^{(t)(r)} = \frac{Q^{(r)(r)}[w^{(t)}]^2 + Q^{(t)(t)}[w^{(r)}]^2 + Q^{(\theta)(\theta)}(-[w^{(t)}]^2 + [w^{(r)}]^2)}{2w^{(t)}w^{(r)}},$$

$$Q^{(t)(\theta)} = \frac{Q^{(r)(\theta)}w^{(t)}}{w^{(r)}}, \quad Q^{(\theta)(\phi)} = \frac{Q^{(r)(\theta)}w^{(\phi)}}{w^{(r)}},$$

$$Q^{(t)(\phi)} = \frac{Q^{(r)(\phi)}w^{(t)}}{w^{(r)}} + \frac{w^{(\phi)}}{2w^{(t)}[w^{(r)}]^2}\{Q^{(t)(t)}[w^{(r)}]^2 - Q^{(r)(r)}[w^{(t)}]^2 + Q^{(\theta)(\theta)}([w^{(t)}]^2 + [w^{(r)}]^2)\},$$

$$Q^{(\phi)(\phi)} = \frac{1}{[w^{(r)}]^2}\Big[Q^{(\theta)(\theta)}([w^{(r)}]^2 + [w^{(\phi)}]^2) + w^{(\phi)}(2Q^{(r)(\phi)}w^{(r)} - Q^{(r)(r)}w^{(\phi)})\Big].$$

Conclusions

- In the pole-dipole case there seems to be only few types of geodesic trajectories in Kerr ($J_z = aE$) that can be followed by an extended body.
- Ohashi-Kyrian-Semerák SSC vanishing hidden momentum feature in pole-dipole approximation can be retrieved in the pole-dipole-quadrupole approximation.
- We failed to find a pole-dipole-(spin induced)quadrupole body setup able to follow a geodesic orbit in Schwarzschild and Kerr.
- Assuming the dipole and the quadrupole components vanish independently on given geodesic trajectories, we get constraints between the mass quadrupole and the the reference vector w^{μ}. This implies that w^{μ} has to dictate the evolution of the mass quadrupole through the constraints.

Thank you for your attention!