An ‘exact’ modified Tolman VII solution

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Introduction

Neutron stars as fundamental physics labs

- Neutron stars (NSs) are unique environments to test fundamental physics, including nuclear physics. They squeeze more than 1.4 $M_\odot$ into a Bogotá city-size volume, providing densities beyond the nuclear saturation limit. Thus they offer natural 'labs' to probe the nuclear EOS.
- The relation between mass (M) and radius (R) of NSs depends on the EOS. In principle, one can probe the EOS by measuring M and R independently [Guver and Ozel (2013); Steiner, Lattimer, and Brown (2010); Lattimer and Steiner (2014)]. For instance, data from the missions *NICER* and *XMM-Newton*, has allowed to estimate M and R of a pulsar to $\sim 5\%$ accuracy [Miller et al. (2021)].
- Other important observables of NSs, which could be useful to test nuclear physics, are the moment of inertia I, to be measured from observations of the double pulsar [Kramer and Wex (2009)]; and the tidal deformability (*Love number* κ_2) which is encoded in GWs from binary NSs mergers [Hinderer (2008); Damour and Nagar (2009); Hinderer et al. (2010)].

Interior solutions for NSs

- To connect NS observables to their interior structure, one must solve Einstein’s equations for a given EOS. Due to the complexity of the field equations, most of the solutions are numerical. However, some analytic solutions exist that are used to ‘mimic’ NSs, e.g., Schwarzschild’s interior solution [Schwarzschild (1916)], Buchdahl [Buchdahl (1967)], and Tolman VII (T-VII) solution [Tolman (1939)]. Analytic NSs solutions are useful to have a better understanding of NSs physics.
- About T-VII, Tolman wrote: *“The dependence of p on r [...] is so complicated that the solution is not a convenient one for physical considerations”* [Tolman (1939)]. The advent of modern computing has allowed further studies of the T-VII solution. It turns out that this solution describes relatively well the interior of realistic NSs [Lattimer and Prakash (2001)].
- Further studies of T-VII: quasi-normal modes and associated universal relations [Tsui and Leung (2005); Tsui, Leung, and Wu (2006)], geometric structure [Neary, Ishak, and Lake (2001); Raghoonundun and Hobill (2015)], tidal Love numbers [Postnikov, Prakash, and Lattimer (2010)], radial stability [Negi and Durgapal (2001); Moustakidis (2017)], extension to scalar-tensor gravity [Sotani and Kokkotas (2018)], trapping of null geodesics [Stuchlík et al. (2021); Stuchlík and Vrba (2021)].

A modified T-VII solution

- Recently Jiang and Yagi, 2019 proposed a modified Tolman VII (MT-VII) solution which seems to describe more accurately the realistic numerical solutions for NSs. The MT-VII introduces a parameter α to allow the energy density to be a quartic function of r. However, after this modification, not all of Einstein's equations are solvable exactly, thus certain approximate relations were proposed.
- Further studies of MT-VII: Analytic I-Love-$\mathcal{C}$ relations [Jiang and Yagi (2020)], radial stability [Posada, Hladík, and Stuchlík (2021)]. In the latter, we found that MT-VII is radially stable in a wide parameter space region $(\mathcal{C}, \alpha)$. However, a further study of MT-VII showed that, for certain values of $(\mathcal{C}, \alpha)$, there appear regions with negative pressure and, consequently, negative tidal deformability. This is in conflict with what is expected for a realistic NS.

Objective

To alleviate the shortcomings of the MT-VII solution, we propose here an 'exact' modified Tolman VII solution (EMT-VII) by solving numerically Einstein's equations for the MT-VII energy density profile. In contrast with MT-VII, our solution shows positive pressure everywhere inside the star, and also a positive tidal Love number. We also provide some constraints based on GW170817.

Tolman’s method for interior solutions

We assume a spherically symmetric matter distribution, with line element,

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

We approximate the internal configuration as a perfect fluid with EMT

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + pg_{\mu\nu},$$

where ϵ is the energy density, p is the pressure and $u^\nu = dx^\nu/d\tau$ is the four-velocity. With these assumptions, Einstein’s equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ read

$$\frac{d}{dr}\left(\frac{e^{-\lambda}-1}{r^2}\right) + \frac{d}{dr}\left(\frac{e^{-\lambda}\nu'}{2r}\right) + e^{-(\lambda+\nu)}\frac{d}{dr}\left(\frac{e^{\nu}\nu'}{2r}\right) = 0\,,$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi p; \qquad \frac{dm}{dr} = 4\pi r^2\epsilon\,,$$

where $m(r) =$ (mass enclosed in the radius r). We can connect $m(r)$ and g_{rr} as

$$e^{-\lambda(r)} \equiv 1 - \frac{2m(r)}{r}\,.$$

We close the ‘Einstein system’ by imposing an equation of state (EOS), i.e., $p = p(\epsilon)$. In a more mathematical approach, Tolman, 1939 chose conveniently e^λ and e^ν, so the system can be integrated analytically. With this approach, Tolman rediscovered several solutions (Schwarzschild’s interior sol., de Sitter, Einstein’s Universe). We focus on the so-called, Tolman VII solution (TVII).

Tolman VII (T-VII) solution

The basic ingredients of the T-VII solution are the following

$$\epsilon_{\text{Tol}} = \epsilon_{\text{c}}(1 - x^2); \quad m_{\text{Tol}} = \frac{M}{2}x^3(5 - 3x^2)\,,$$

$$p_{\text{Tol}} = \frac{\epsilon_{\text{c}}}{15}\left[\sqrt{\frac{12e^{-\lambda_{\text{Tol}}}}{\mathcal{C}}}\tan\phi_{\text{Tol}} - (5 - 3x^2)\right]; \quad \phi_{\text{Tol}} = C_2^{\text{Tol}} - \frac{1}{2}\log\left(x^2 - \frac{5}{6} + \sqrt{\frac{5e^{-\lambda_{\text{Tol}}}}{8\pi\epsilon_{\text{c}}R^2}}\right),$$

$$e^{-\lambda_{\text{Tol}}} = 1 - \frac{8\pi}{15}\epsilon_{\text{c}}R^2x^2(5 - 3x^2); \quad e^{\nu_{\text{Tol}}} = C_1^{\text{Tol}}\cos^2\phi_{\text{Tol}}\,.$$

C_1^{Tol} and C_2^{Tol} are integration constants, $x \equiv r/R$ where R is the stellar radius, ϵ_{c} is the central energy density, and $\mathcal{C} \equiv M/R$ is the compactness. Important constraints:

- $p_{\text{c}} \to \infty$ when $\mathcal{C} = 0.386$.
- $c_{\text{s}} < 1$ for $\mathcal{C} < 0.270$.
- DEC is valid for $\mathcal{C} < 0.3351$.

Modified Tolman VII (MT-VII) solution

Jiang and Yagi, 2019 (JY) proposed a modification to the T-VII energy density profile

$$\epsilon_{\text{mod}} = \epsilon_{\text{c}} \left[1 - \alpha x^2 + (\alpha - 1) x^4\right]; \quad e^{-\lambda_{\text{mod}}} = 1 - 8\pi\epsilon_{\text{c}}(Rx)^2 \left[\frac{1}{3} - \frac{\alpha}{5}x^2 + \frac{(\alpha-1)}{7}x^4\right].$$

The g_{rr} metric component, and p, can be found analytically as

$$\tilde{p}_{\text{mod}} = \frac{1}{8\pi}\left[e^{-\lambda_{\text{mod}}}\left(\frac{\nu'_{\text{mod}}}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}\right],$$

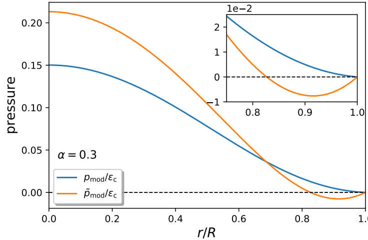
For the MT-VII ansatz, it is not possible to obtain an exact solution for g_{tt}. Moreover, $\tilde{p}_{\text{mod}}$ gives $\tilde{p}_{\text{c}}$ around 20% off from numerical results, and also becomes negative near the surface. Thus, JY proposed the following approximate expressions for e^{ν} and $p(r)$

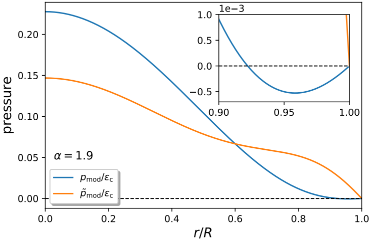
$$e^{\nu_{\text{mod}}} = C_1^{\text{mod}} \cos^2\phi_{\text{mod}}; \quad \phi_{\text{mod}} = C_2^{\text{mod}} - \frac{1}{2}\log\left(x^2 - \frac{5}{6} + \sqrt{\frac{5e^{-\lambda_{\text{Tol}}}}{8\pi R^2\epsilon_{\text{c}}}}\right),$$

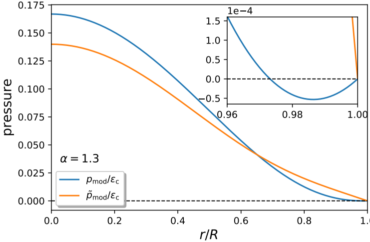
$$\frac{p_{\text{mod}}(x)}{\epsilon_{\text{c}}} = \left(\frac{e^{-\lambda_{\text{Tol}}}}{10\pi\epsilon_{\text{c}}R^2}\right)^{1/2} \tan\phi_{\text{mod}} + \frac{1}{15}(3x^2 - 5) + \frac{6(1-\alpha)}{16\pi\epsilon_{\text{c}}R^2(10-3\alpha) - 105}.$$

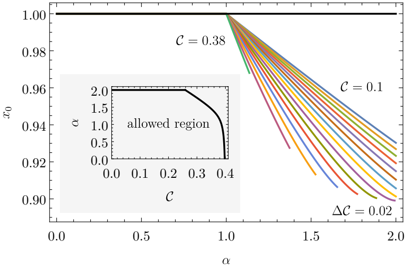
- Parameters of the MT-VII solution: $(\mathcal{C}, R, \alpha)$ [or, (M, R, α)]. These parameters are related by the condition $m(R) = M$, such that $\epsilon_{\text{c}} = 105\mathcal{C}/8\pi R^2(10 - 3\alpha)$.
- Allowed values of $\alpha \in [0, 2]$
- I-Love-$\mathcal{C}$ [Jiang and Yagi (2020)], radial stability [Posada, Hladík, and Stuchlík (2021)].

Analysis of the MT-VII solution



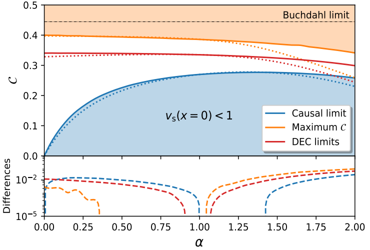




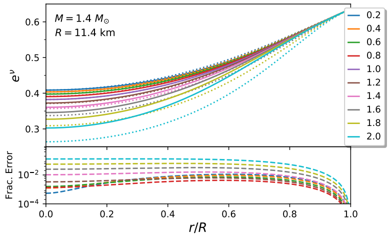


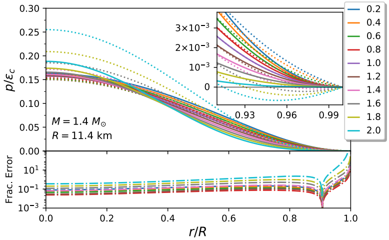
‘Exact’ modified Tolman VII (EMT-VII) solution

Considering the drawbacks of the MT-VII model, we propose an ‘exact’ MT-VII (EMT-VII) solution, by solving numerically Einstein’s equations for g_{tt} and $p(r)$, given the MT-VII energy density $\epsilon_{\rm mod}$ [Posada, Hladík, and Stuchlík (2022)].



Constraints on the compactness $\mathcal{C}$, for $\alpha \in [0, 2]$, predicted by the EMTVII (solid lines) and MTVII (dotted lines) solutions. Blue lines: causality, $v_{\rm s}(x=0) \leq 1$. Orange lines: maximum $\mathcal{C}$ where $p_{c\to\infty}$. Red lines: DEC limits. Bottom panel: fractional error.

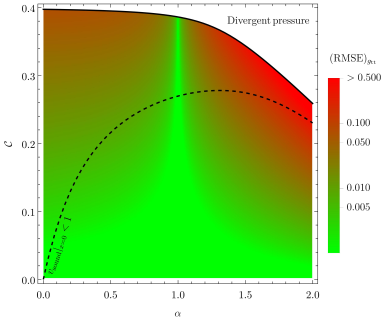




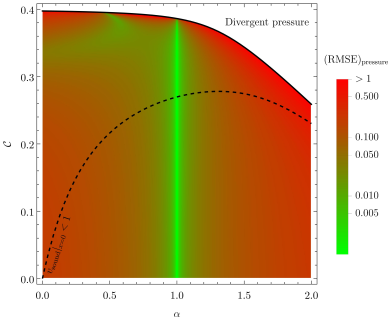
EMT-VII *vs* MT-VII

The results presented, so far, are specific to one configuration. We show here the differences in the whole $(\mathcal{C}, \alpha)$ parameter space (PS) by computing the root-mean-square error (RMSE) as

$$\text{(RMSE)} = \sqrt{\frac{\int_0^R \left[y^{\text{(EMTVII)}} - y^{\text{(MTVII)}}\right]^2 \, dr}{\int_0^R y^{\text{(EMTVII)}]^2} \, dr}}.$$



The RMSE, of the metric component g_{tt}, for the EMTVII and MTVII solutions in the PS $(\alpha, \mathcal{C})$.



The RMSE, of the radial pressure profile, for the EMTVII and MTVII solutions in the PS $(\alpha, \mathcal{C})$.

Tidal Love number

- The ('electric-type') Love number k_2, or alternatively the tidal deformability Λ, provides a connection between the tidal fields ε_{ij} and the induced quadrupole moments Q_{ij} [Hinderer (2008); Damour and Nagar (2009); Binnington and Poisson (2009); Poisson (2021)]

$$Q_{ij} = -\frac{2\kappa_2 R^5}{3}\varepsilon_{ij} \equiv -\Lambda\varepsilon_{ij}.$$

It is conventional to introduce $\bar{\Lambda} = \Lambda/M^5 = 2\kappa_2/(3\mathcal{C}^5)$, as employed in the context of the I-Love-Q relations for NSs [Yagi and Yunes (2013)].

$$\kappa_2 = \frac{8}{5}(1-2\mathcal{C})^2\mathcal{C}^5\left[2\mathcal{C}(h_R-1)-h_R+2\right]\Big\{2\mathcal{C}\big[4(h_R+1)\mathcal{C}^4 + (6h_R-4)\mathcal{C}^3 + (26-22h_R)\mathcal{C}^2 + 3(5h_R-8)\mathcal{C} - 3h_R + 6\big] + 3(1-2\mathcal{C})^2\left[2\mathcal{C}(h_R-1)-h_R+2\right]\log(1-2\mathcal{C})\Big\}^{-1},$$

where $h_R = [(r/H)dH/dr]_{r=R}$.

- The Love number is strongly sensitive to $\mathcal{C}$.
- Tidal Love numbers have been computed for various EOS for NSs [Hinderer (2008); Damour and Nagar (2009); Hinderer et al. (2010); Postnikov, Prakash, and Lattimer (2010)].

Stationary perturbation equation

Even-parity, stationary perturbations of a barotropic star are characterised by the following:

1. Metric perturbations reduce to two functions $H = H_0 = H_2$, and K
2. Fluid perturbations are represented by a thermodynamic quantity h: $\delta h = \delta p/(p+\epsilon)$
3. Both are related via $\delta h = -\frac{1}{2}H$

Metric perturbations satisfy the following ODE (Lindblom, Mendell, and Ipser, 1997)

$$\frac{d^2 H}{dr^2} + C_1(r)\frac{dH}{dr} + C_0(r)H = 0,$$

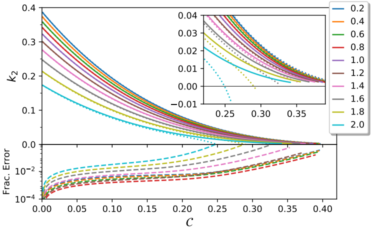
with coefficients

$$C_1(r) = \frac{2}{r} + e^{\lambda}\left[\frac{2m(r)}{r^2} + 4\pi r(p-\epsilon)\right],$$

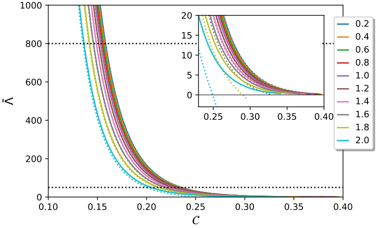
$$C_0(r) = e^{\lambda}\left[-\frac{l(l+1)}{r^2} + 4\pi(\epsilon+p)\frac{d\epsilon}{dp} + 4\pi(5\epsilon+9p)\right] - \left(\frac{d\nu}{dr}\right)^2.$$

What about Love? Take the equilibrium configuration and integrate the perturbation equation for H from the center outwards. Then, determine the value of the logarithmic derivative at the surface.

Love number of EMT-VII



Tidal Love number k_2, as a function of $\mathcal{C}$, for the EMTVII (solid) and MTVII (dotted) solutions, for $\alpha \in [0, 2]$.



$\bar{\Lambda}$ vs $\mathcal{C}$, for the EMTVII (solid) and MTVII (dotted) solutions, for $\alpha \in [0, 2]$. We include the constraints for $\bar{\Lambda}$ from GW170817 [Abbott et al., 2019].

- For a NS with $M = 1.4\ M_\odot$ and $R = 11.4$ km, frac. errors go from $\sim 1\%$ for $\alpha \to 0$, up to $\sim 10\%$ for $\alpha = 2$.
- In the regime considered by (JY), $\alpha \in [0.4, 1.4]$, $\mathcal{C} \in [0.05, 0.35]$, diff. in Love are relatively low between MT-VII and EMT-VII. However, for $\alpha = 1.6$ with $\mathcal{C} > 0.25$, the differences grow.
- From our EMT-VII model, for $\alpha = 0.2, \to \mathcal{C} = 0.235$, while for $\alpha = 2, \to \mathcal{C} = 0.205$. This small diff. would give a maximum GW frequency $f_{\max} \sim \sqrt{GM/(\pi^2 R^3)}$, around 18% larger for the NS with $\alpha = 0.2$.

Final Remarks

- Although the MT-VII model, for certain values of $(\mathcal{C}, \alpha)$, fits well with the realistic EOS for NSs, we found that it predicts regions with negative pressure near the surface. As a consequence, the MT-VII model predicts a negative tidal Love number for certain configurations.
- We propose here the semi-analytical EMT-VII solution, by solving Einstein's equations for the quartic energy density profile introduced by Jiang and Yagi, 2019. Our solution shows positive pressure, and positive tidal deformability, in the whole allowed regime of $(\mathcal{C}, \alpha)$, which is consistent with what is expected for realistic NSs.
- Some ideas for further studies of the EMT-VII model: stability, trapping of null geodesics, extension to slow rotation, I-Love-$\mathcal{C}$ relations.