Astrophysical black holes embedded in organized magnetic fields

Exact electrovacuum solutions with axial symmetry and stationarity

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The spacetime metric

$$\mathrm{d}s^2 = f^{-1}\left[e^{2\gamma}\left(\mathrm{d}z^2 + \mathrm{d}\rho^2\right) + \rho^2\,\mathrm{d}\phi^2\right] - f\,(\,\mathrm{d}t - \omega\,\mathrm{d}\phi)^2\,,$$

with f, ω, and γ functions of z and ρ only.

We consider coupled Einstein-Maxwell equations:

electrovacuum case with a black hole,

axial symmetry, stationarity,

not necessarilly asymptotically flat.

- Kramer D., Stephani H., MacCallum M., and Herlt E., *Exact Solutions of the Einstein's Field Equations* (Deutscher Verlag der Wissenschaften, Berlin 1980)
- Alekseev G. A., Garcia A. A., Phys. Rev. D **53**, 1853 (1996)
- Ernst F. J., and Wild W. J., J. Math. Phys. **12**, 1845 (1976)
- Hiscock W. A., J. Math. Phys. **22**, 1828 (1988)
- Karas V., Vokrouhlický D., J. Math. Phys. **32**, 714 (1990)

The spacetime metric

Finding the three metric functions:

- Standard approach: $g_{\mu\nu} \to \Gamma^{\mu}_{\nu\lambda} \to R^{\alpha}_{\beta\gamma\delta} \to G_{\mu\nu}$
- Exterior calculus: $e^{\mu}_{(\lambda)} \to \omega_{\mu\nu}\Omega_{\mu\nu} \to R^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}\hat{\delta}} \to G_{\hat{\mu}\hat{\nu}}$
- Variation principle: $\mathcal{L} = -\frac{1}{2}\rho f^{-2}\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} f + \frac{1}{2}\rho^{-1}f^2\boldsymbol{\nabla}\omega\cdot\boldsymbol{\nabla}\omega$

$$\boldsymbol{\nabla}\cdot\left(\rho^{-1}\boldsymbol{e_\phi}\times\boldsymbol{\nabla}\varphi\right) = 0 \;\forall\varphi \equiv \varphi(\rho, z)$$

Vacuum field equations (without electromagnetic field):

$$\begin{aligned} & f\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} f - \rho^{-2}f^4\boldsymbol{\nabla}\omega\cdot\boldsymbol{\nabla}\omega, \\ & \boldsymbol{\nabla}\cdot\left(\rho^{-2}f^2\boldsymbol{\nabla}\omega\right) = 0. \end{aligned}$$

Define functions $\varphi(\rho, z)$, $\omega(\rho, z)$ by

$$\begin{aligned} \rho^{-1}f^2\boldsymbol{\nabla}\omega &= \boldsymbol{e_\phi}\times\boldsymbol{\nabla}\varphi, \\ f^{-2}\boldsymbol{\nabla}\varphi &= -\rho^{-1}\boldsymbol{e_\phi}\times\boldsymbol{\nabla}\omega \qquad /\boldsymbol{\nabla}\cdot \end{aligned}$$

Ernst equation

Equation for φ is

$$\boldsymbol{\nabla} \cdot \left(f^{-2} \boldsymbol{\nabla} \varphi\right) = 0.$$

Define $\mathcal{E} \equiv f + \Im \varphi$. Then, both field equations can be written in the form:

$$(\Re \mathcal{E}) \boldsymbol{\nabla}^2 \mathcal{E} = \boldsymbol{\nabla} \mathcal{E} \cdot \boldsymbol{\nabla} \mathcal{E}.$$

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Inclusion of electromagnetic field:

$$\mathcal{L}'=\mathcal{L}+2\rho f^{-1}A_0\left(\boldsymbol{\nabla}A\right)^2-2\rho^{-1}f\left(\boldsymbol{\nabla}A_3-\omega\boldsymbol{\nabla}A_0\right)^2$$

Variation principle for f, ω, A_0, A_3. Define $\Phi\equiv\Phi(A_0,A_3)$, $\mathcal{E}\equiv f-|\Phi|^2+\Im\varphi$:

$$\left(\Re\mathcal{E}+|\Phi|^2\right)\boldsymbol{\nabla}^2\mathcal{E}=\left(\boldsymbol{\nabla}\mathcal{E}+2\bar{\Phi}\boldsymbol{\nabla}\Phi\right)\cdot\boldsymbol{\nabla}\mathcal{E},$$
$$\left(\Re\mathcal{E}+|\Phi|^2\right)\boldsymbol{\nabla}^2\Phi=\left(\boldsymbol{\nabla}\mathcal{E}+2\bar{\Phi}\boldsymbol{\nabla}\Phi\right)\cdot\boldsymbol{\nabla}\Phi.$$

Generating "new" solutions

Theorem. Let $(\Phi, \mathcal{E}, \gamma_{\alpha\beta})$ be a solution of Einstein-Maxwell electrovaccum eqs. with anisotropic Killing vector field. Then there is another solution $(\Phi', \mathcal{E}', \gamma'_{\alpha\beta})$, related to the old one by transformation

$$
\begin{aligned}
\mathcal{E}' &= \alpha\bar{\alpha}\mathcal{E}, \quad \Phi' = \alpha\Phi, \quad \ldots\text{dual rotation, } {}^\star F_{\mu\nu} \to \sqrt{\alpha/\bar{\alpha}}\,{}^\star F_{\mu\nu}, \\
\mathcal{E}' &= \mathcal{E} + \Im b, \quad \Phi' = \Phi, \quad \ldots\text{calibration, no change in } F_{\mu\nu}, \\
\mathcal{E}' &= \mathcal{E} - 2\bar{\beta}\Phi - \beta\bar{\beta}, \quad \Phi' = \Phi + \beta, \quad \ldots\text{calibration} \ldots, \\
\mathcal{E}' &= \mathcal{E}(1 + \Im c\mathcal{E})^{-1}, \quad \Phi' = (1 + \Im c\mathcal{E})^{-1}, \\
\mathcal{E}' &= \mathcal{E}\underbrace{(1 - 2\bar{\gamma}\Phi - \gamma\bar{\gamma}\mathcal{E})}_{\Lambda = 1 - B_0\Phi - \frac{1}{4}B_0^2\mathcal{E}}{}^{-1}, \quad \Phi' = (\Phi + \gamma\mathcal{E})(1 - 2\bar{\gamma}\Phi - \gamma\bar{\gamma}\mathcal{E})^{-1}.
\end{aligned}
$$

$$\mathcal{E} \to \mathcal{E}' = \Lambda^{-1}\mathcal{E}, \qquad f \to f' = |\Lambda|^{-2} f, \qquad \omega \to \omega' = ?,$$

$$\Phi \to \Phi' = \Lambda^{-1}(\Phi - \tfrac{1}{2}B_0\mathcal{E}), \ \boldsymbol{\nabla}\omega' = |\Lambda|^2\boldsymbol{\nabla}\omega + \rho f^{-1}(\bar{\Lambda}\boldsymbol{\nabla}\Lambda - \Lambda\boldsymbol{\nabla}\bar{\Lambda}).$$

Examples

Example 1. Minkowski spacetime $\rightarrow$ Melvin universe:

$$\mathrm{d}s^2 = \left[\,\mathrm{d}z^2 + \,\mathrm{d}\rho^2 - \,\mathrm{d}t^2\right] + \rho^2\,\mathrm{d}\phi^2.$$

$$f = -\rho^2, \quad \omega = 0, \quad \Phi = 0, \quad \mathcal{E} = -\rho^2, \quad \varphi(\omega) = 0.$$

$$f' = -\Lambda^{-2}\rho^2, \quad \omega' = 0, \quad \Phi' = \tfrac{1}{2}\Lambda^{-1}B_0\rho^2,$$

$$B_z = \Lambda^{-2}B_0, \quad B_\rho = B_\phi = 0,$$

$$\mathrm{d}s^2 = \Lambda^2\left[\,\mathrm{d}z^2 + \,\mathrm{d}\rho^2 - \,\mathrm{d}t^2\right] + \Lambda^{-2}\rho^2\,\mathrm{d}\phi^2.$$

Gravity of the magnetic field in balance with the Maxwell pressure. Cylindrical symmetry along z-axis.

Examples

Example 2. Schwarzschild BH $\rightarrow$ Schwarzschild-Melvin BH:

$$\mathrm{d}s^2 = \left[\left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 - \left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + r^2\,\mathrm{d}\theta^2\right] + r^2 \sin^2\theta\,\mathrm{d}\phi^2,$$

$$f = -r^2 \sin^2\theta, \quad \omega = 0, \quad \rho = \sqrt{r^2 - 2Mr}\,\sin\theta,$$

$$B_r = \Lambda^{-2} B_0 \cos\theta, \quad B_\theta = -\Lambda^{-2} B_0 (1 - 2M/r) \sin\theta,$$

$$\mathrm{d}s^2 = \Lambda^2 \left[\quad \ldots \quad\right] + \Lambda^{-2} r^2 \sin^2\theta\,\mathrm{d}\phi^2.$$

$B_0 = 0$ ... Schwarzschild solution,
$r/M \rightarrow \infty$ ... Melvin solution,

$|B_0 M| \ll 1$ ... Wald's test field in the region $2M \ll r \ll B_0^{-1}$.

Examples

Example 3. Magnetized Kerr-Newman BH:

$$\begin{aligned} g &= |\Lambda|^2\Sigma\left(\Delta^{-1}\,\mathrm{d}r^2 + \,\mathrm{d}\theta^2 - \Delta A^{-1}\,\mathrm{d}t^2\right) \\ &\quad + |\Lambda|^{-2}\Sigma^{-1} A\sin^2\theta\,(\,\mathrm{d}\phi - \omega\,\mathrm{d}t)^2\,, \end{aligned}$$

$\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2 + e^2$, $A = (r^2+a^2)^2 - \Delta a^2\sin^2\theta$ are functions from the Kerr-Newman metric.
$\Lambda = 1 + \beta\Phi - \frac{1}{4}\beta^2\mathcal{E}$ is given in terms of the Ernst complex potentials $\Phi(r,\theta)$ and $\mathcal{E}(r,\theta)$:

$$\begin{aligned} \Sigma\Phi &= ear\sin^2\theta - \Im e\left(r^2+a^2\right)\cos\theta, \\ \Sigma\mathcal{E} &= -A\sin^2\theta - e^2\left(a^2 + r^2\cos^2\theta\right) \\ &\quad + 2\Im a\left[\Sigma\left(3-\cos^2\theta\right) + a^2\sin^4\theta - re^2\sin^2\theta\right]\cos\theta. \end{aligned}$$

Examples

The electromagnetic field in terms of orthonormal LNRF components,

$$
\begin{aligned}
H_{(r)} + \mathrm{i}E_{(r)} &= A^{-1/2} \sin^{-1}\theta \, \Phi'_{,\theta}, \\
H_{(\theta)} + \mathrm{i}E_{(\theta)} &= -\left(\Delta/A\right)^{1/2} \sin^{-1}\theta \, \Phi'_{,r},
\end{aligned}
$$

where $\Phi'(r,\theta) = \Lambda^{-1}\left(\Phi - \frac{1}{2}\beta\mathcal{E}\right)$.

The horizon: $r \equiv r_+ = 1 + \sqrt{(}1 - a^2 - e^2)$, independent of β. As in the non-magnetized case, the horizon exists only for $a^2 + e^2 \leq 1$.

Examples

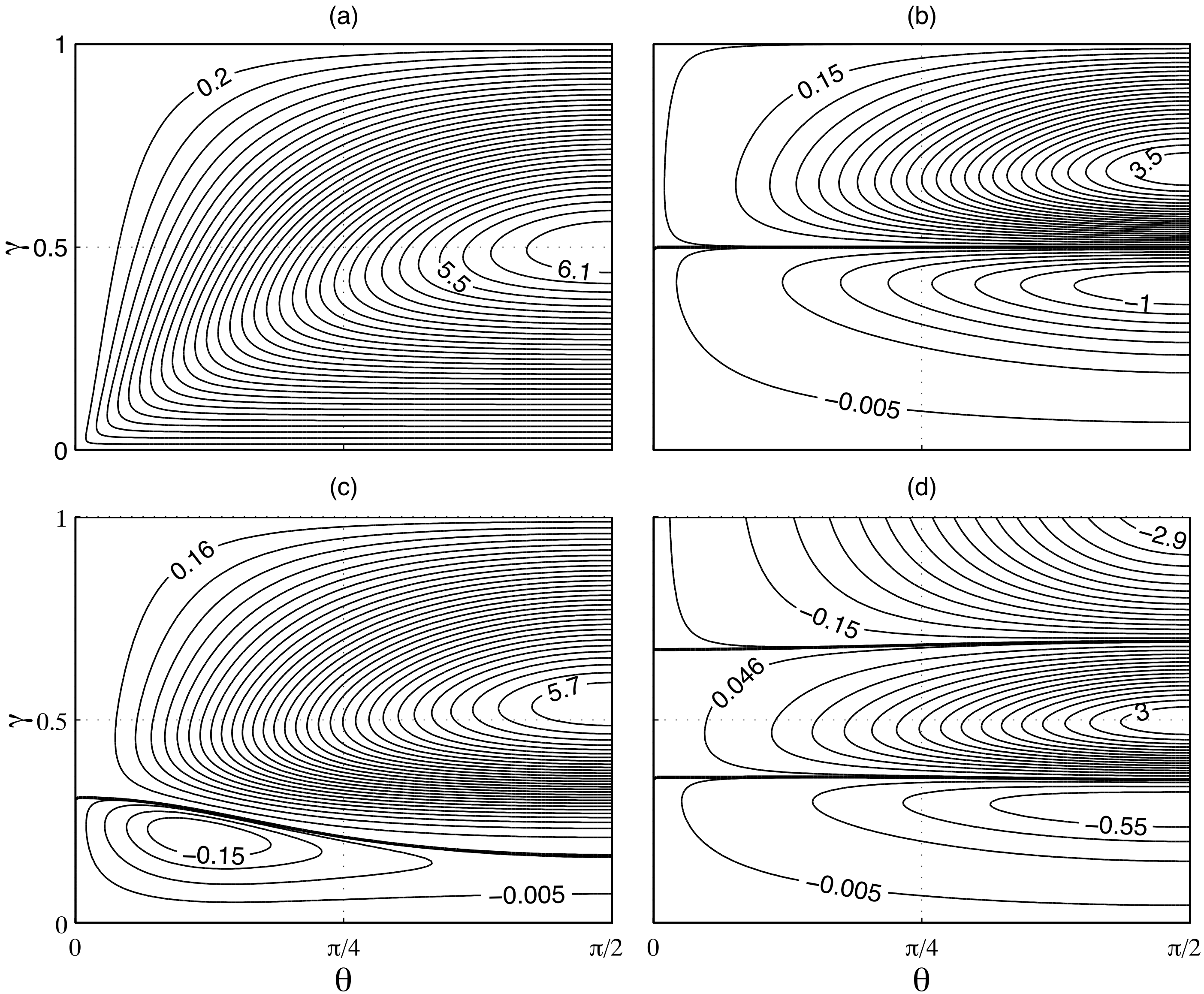
The range of angular coordinates *versus* the problem of conical singularity: $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi|\Lambda_0|^2$, where

$$|\Lambda_0|^2 \equiv |\Lambda(\sin\theta = 0)|^2 = 1 + \tfrac{3}{2}\beta^2 e^2 + 2\beta^3 ae + \beta^4 \left(\tfrac{1}{16}e^4 + a^2\right) .$$

The total electric charge $Q_{\rm H}$ and the magnetic flux $\Phi_{\rm m}(\theta)$ across a cap in axisymmetric position on the horizon (with the edge defined by $\theta = {\rm const}$):

$$\begin{aligned} Q_{\rm H} &= -|\Lambda_0|^2\, \Im{\rm m}\, \Phi'\,(r_+, 0)\,, \\ \Phi_{\rm m} &= 2\pi|\Lambda_0|^2\, \Re{\rm e}\, \Phi'\left(r_+, \bar\theta\right)\Big|_{\bar\theta=0}^{\theta}. \end{aligned}$$

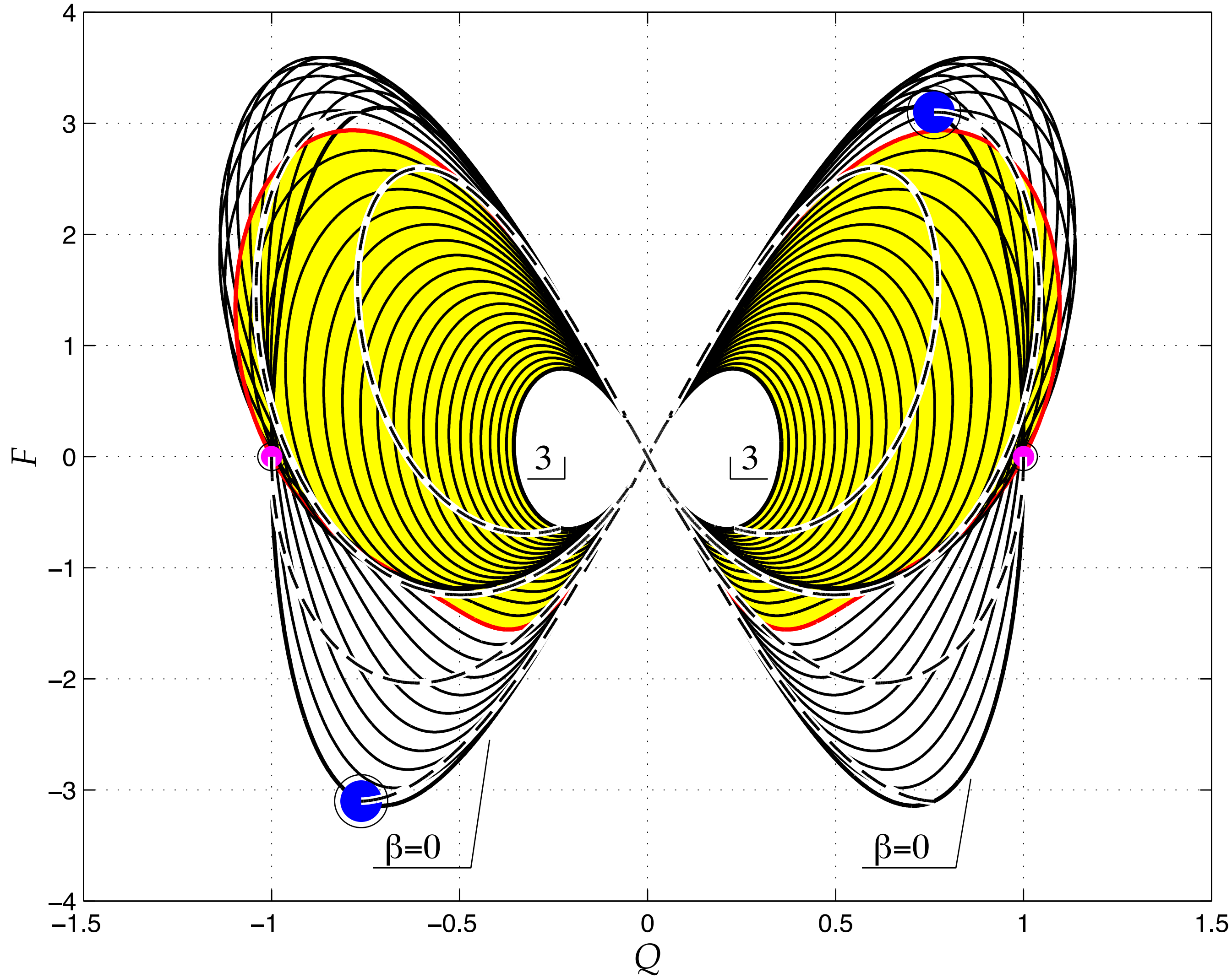
Magnetic flux across a cap $r = r_+$, $\theta = \mathrm{const}$



(a) $a = e = 0$; (b) $a = 1$, $e = 0$; (c) $a = 0.2$, $e = 0$; (d) $a = -e = 1/\sqrt{2}$.

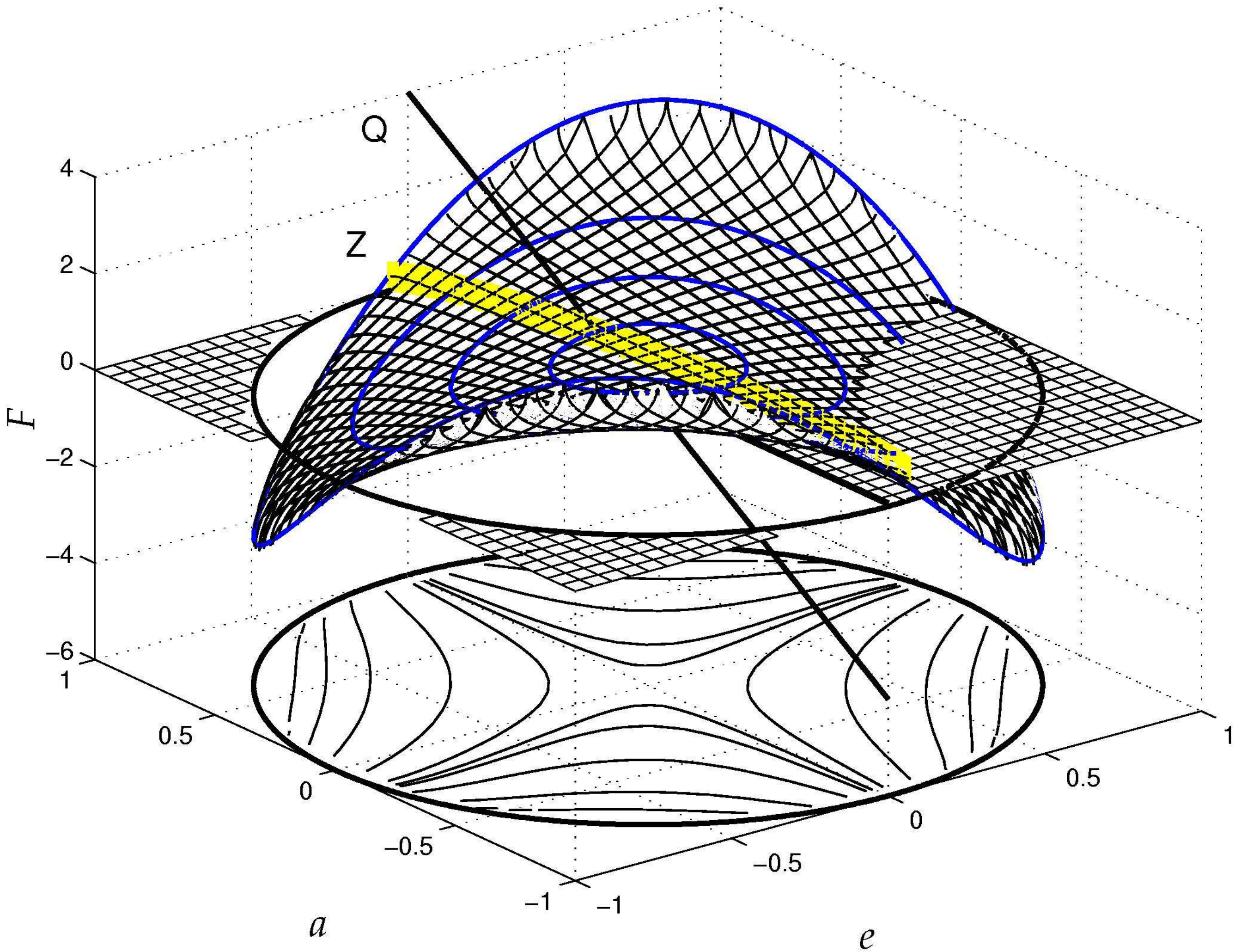
Here, $\gamma \equiv (1+\beta)^{-1}$, $\beta \equiv B_0 M$.

Extreme magnetized Kerr-Newman BH



Solid curves correspond to a fixed value of β in the range $\langle 0, 3\rangle$; $\beta = 0$ is the case of Kerr-Newman black hole with no external magnetic field. The lines of constant ratio a/e and varying β are also plotted (dashed).

The flux $F(a, e)$ across the whole hemisphere $\theta=\pi/2$



The surface is defined on the circle $a^2 + e^2 \leq 1$ for a fixed value of $\beta = 0.05$. Four circles of $\sqrt(a^2 + e^2) = 0.25$, 0.5, 0.75, and 1.0 are shown. The shaded band on the surface, denoted by “Z”, indicates where the total electric charge is zero. $Q(a, e = 0)$ does not generally vanish and its graph is shown by solid curve “Q”.

Thank you!

Assume $\mathcal{E} \equiv \mathcal{E}(\Phi)$ to be an analytic function $\rightarrow$

$$\left(\Re\mathcal{E} + \Phi^2\right) \frac{\mathrm{d}^2\mathcal{E}}{\mathrm{d}\Phi^2} \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\Phi = 0.$$

Assume further a linear relation,

$$\mathcal{E} = 1 - 2\frac{\Phi}{q}, \qquad q \in \mathrm{C}$$

New variable ξ:

$$\mathcal{E} \equiv \frac{\xi - 1}{\xi + 1}, \qquad \Phi = \frac{q}{\xi + 1},$$

$$[\xi\bar{\xi} - (1 - q\bar{q})]\boldsymbol{\nabla}^2\xi = 2\bar{\xi}\boldsymbol{\nabla}\xi \cdot \boldsymbol{\nabla}\xi.$$

Generating "new" solutions

$$\xi_0 \to \xi = (1 - q\bar{q})\xi_0,$$

$$[\xi_0\bar{\xi}_0 - 1]\boldsymbol{\nabla}^2\xi_0 = 2\bar{\xi}_0\boldsymbol{\nabla}\xi_0 \cdot \boldsymbol{\nabla}\xi_0.$$

i.e.

$$(\Re\mathcal{E}_0)\boldsymbol{\nabla}^2\mathcal{E}_0 = \boldsymbol{\nabla}\mathcal{E}_0 \cdot \boldsymbol{\nabla}\mathcal{E}_0, \qquad \mathcal{E}_0 \equiv \frac{\xi_0 - 1}{\xi_0 + 1}.$$

$\mathcal{E}_0$ is an "old" vacuum solution.